A general property of Lemoine point

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1. Introduction

It has been demonstrated[1] that Lemoine point is in the line that connects Vecten points X(485) and X(486). Also, it can be demonstrated that Lemoine point is in the line that connects the Fermat points X(13) and X(14). Furthermore, it is collinear with the Napoleon points X(17) and X(18). Then, it comes the following question: Does Lemoine point belong to each line that connects a pair of points P_E and P_I, whose procedure to obtain them is analogous to the procedure to obtain the pairs of points mentioned before? [2]The answer is yes.

2. Demonstration

Given the ABC triangle and the trilinear coordinates:

Lemoine point, P_L $\quad \sin A : \sin B : \sin C$

Point P_I $\quad$ $$\frac{1}{\sin(A-\lambda)} : \frac{1}{\sin(B-\lambda)} : \frac{1}{\sin(C-\lambda)}$$

Point P_E $\quad$ $$\frac{1}{\sin(A+\lambda)} : \frac{1}{\sin(B+\lambda)} : \frac{1}{\sin(C+\lambda)}$$

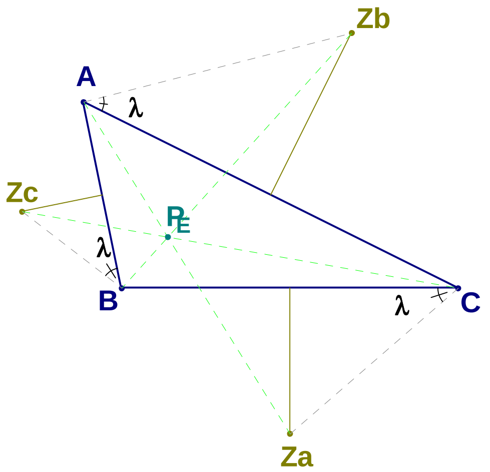
P_E represents the point of concurrence of the three lines drawn from each vertex A, B, C, to the points Z_a, Z_b, Z_c, that are out of the triangle and over the perpendicular bisectors of the sides of the triangle. The angle λ is made by each side of the triangle AC, BA, CB, with the lines AZ_b, BZ_c, CZ_a. The point P_I is obtained in a similar way but the difference is that points Z_a, Z_b, Z_c, have been built towards the opposite direction. So, we can prove that:

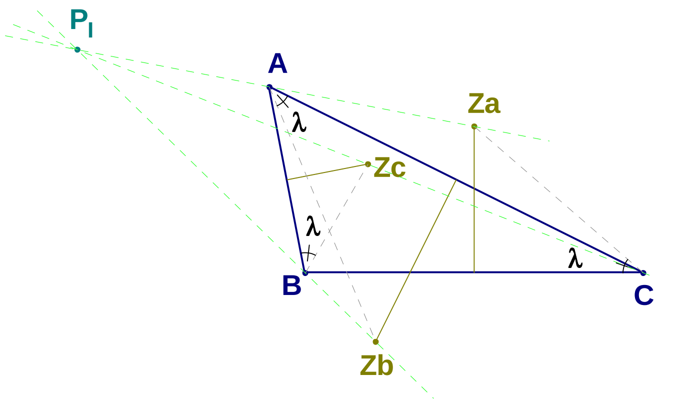
- $\lambda = \frac{\pi}{3}$, it means that P_E and P_I are Fermat points.
- $\lambda = \frac{\pi}{4}$, it means that P_E and P_I are Vecten points.

[1] After Darij Grinberg informed of this fact, the Naval Engineer José María Pedret published a demonstration in the problem number 163 in the website **http://www.personal.us.es/rbarroso/trianguloscabri/** edited by Ricardo Barroso Campos, Professor in Mathematics at University of Seville, who made a few comments about this demonstration. This comments and the method displayed by Pedret, have been very important to develop this general property.

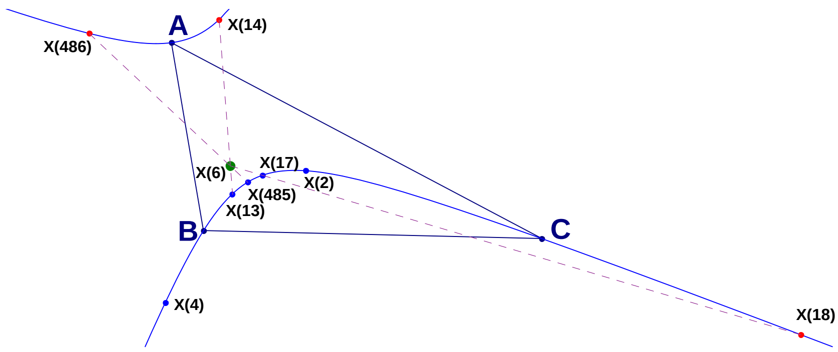
[2] Look at the same website, the extended version of the problem number 13 written by David Benitez Mojica, Professor at University of Coahuila. I show my gratitude to him for his geometrical contribution.

- $\lambda = \frac{\pi}{6}$, it means that P_E and P_I are Napoleon points.
- $\lambda = 0$, it means that P_E and P_I coincides in the centroid X(2).
- $\lambda = \frac{\pi}{2}$, it means that P_E and P_I coincides in the orthocenter X(4) and that Z_a, Z_b, Z_c go to the infinity.





It is interesting to emphasize the fact that the variations of angular parameter λ make the points P_E and P_I have as geometrical place a hyperbola[3], which is equilateral when it goes along the three vertices A, B, C, and the orthocenter



Then, if P_L, P_I, and P_E are collinear, it has to be true the following determinant:

$$\begin{vmatrix} \sin A & \sin B & \sin C \\ \dfrac{1}{\sin(A-\lambda)} & \dfrac{1}{\sin(B-\lambda)} & \dfrac{1}{\sin(C-\lambda)} \\ \dfrac{1}{\sin(A+\lambda)} & \dfrac{1}{\sin(B+\lambda)} & \dfrac{1}{\sin(C+\lambda)} \end{vmatrix} = 0$$

If we start to work it out from the last line, we obtain three terms

$$\frac{1}{\sin(A+\lambda)} \begin{vmatrix} \sin B & \sin C \\ \dfrac{1}{\sin(B-\lambda)} & \dfrac{1}{\sin(C-\lambda)} \end{vmatrix} - \frac{1}{\sin(B+\lambda)} \begin{vmatrix} \sin A & \sin C \\ \dfrac{1}{\sin(A-\lambda)} & \dfrac{1}{\sin(C-\lambda)} \end{vmatrix} +$$

[3]Established by Professor David Benítez Mojica

$$\frac{1}{\sin(C+\lambda)}\begin{vmatrix} \sin A & \sin B \\ \dfrac{1}{\sin(A-\lambda)} & \dfrac{1}{\sin(B-\lambda)} \end{vmatrix} =$$

The first term

$$\frac{1}{\sin(A+\lambda)}\begin{vmatrix} \sin B & \sin C \\ \dfrac{1}{\sin(B-\lambda)} & \dfrac{1}{\sin(C-\lambda)} \end{vmatrix} = \frac{1}{\sin(A+\lambda)}\left[\frac{\sin B}{\sin(C-\lambda)} - \frac{\sin C}{\sin(B-\lambda)}\right] =$$

$$\frac{\sin B\sin(B-\lambda) - \sin C\sin(C-\lambda)}{\sin(A+\lambda)\sin(B-\lambda)\sin(C-\lambda)} =$$

if we bear in mind that $\boxed{\sin\alpha\sin\beta = -\frac{1}{2}[\cos(\alpha+\beta)-\cos(\alpha-\beta)]}$

$$\frac{-\frac{1}{2}\Big[\cos(B+B-\lambda)-\cos(B-B+\lambda)\Big] + \frac{1}{2}\Big[\cos(C+C-\lambda)-\cos(C-C+\lambda)\Big]}{\sin(A+\lambda)\sin(B-\lambda)\sin(C-\lambda)} =$$

$$\frac{\cos(2C-\lambda)-\cos(2B-\lambda)}{2\sin(A+\lambda)\sin(B-\lambda)\sin(C-\lambda)} \qquad \text{(I)}$$

if we continue solving it in a similar way, the second term is

$$-\frac{1}{\sin(B+\lambda)}\begin{vmatrix} \sin A & \sin C \\ \dfrac{1}{\sin(A-\lambda)} & \dfrac{1}{\sin(C-\lambda)} \end{vmatrix} = \frac{\cos(2A-\lambda)-\cos(2C-\lambda)}{2\sin(B+\lambda)\sin(A-\lambda)\sin(C-\lambda)} \qquad \text{(II)}$$

and the third is

$$\frac{1}{\sin(C+\lambda)}\begin{vmatrix} \sin A & \sin B \\ \dfrac{1}{\sin(A-\lambda)} & \dfrac{1}{\sin(B-\lambda)} \end{vmatrix} = \frac{\cos(2B-\lambda)-\cos(2A-\lambda)}{2\sin(C+\lambda)\sin(A-\lambda)\sin(B-\lambda)} \qquad \text{(III)}$$

Now we write a common denominator for the terms I, II, III

$$\frac{\sin(A-\lambda)\sin(B+\lambda)\sin(C+\lambda)\big[\cos(2C-\lambda)-\cos(2B-\lambda)\big]}{2\sin(A+\lambda)\sin(B+\lambda)\sin(C+\lambda)\sin(A-\lambda)\sin(B-\lambda)\sin(C-\lambda)} \qquad \text{(IV)}$$

$$\frac{\sin(B-\lambda)\sin(A+\lambda)\sin(C+\lambda)\big[\cos(2A-\lambda)-\cos(2C-\lambda)\big]}{2\sin(A+\lambda)\sin(B+\lambda)\sin(C+\lambda)\sin(A-\lambda)\sin(B-\lambda)\sin(C-\lambda)} \qquad \text{(V)}$$

$$\frac{\sin(C-\lambda)\sin(A+\lambda)\sin(B+\lambda)\big[\cos(2B-\lambda)-\cos(2A-\lambda)\big]}{2\sin(A+\lambda)\sin(B+\lambda)\sin(C+\lambda)\sin(A-\lambda)\sin(B-\lambda)\sin(C-\lambda)} \qquad \text{(VI)}$$

By the way, the sum of the numerators IV, V VI, has to be 0.

We have to bear in mind these expressions

$S_1 = \cos A\cos B\sin C\sin^2\lambda\cos\lambda$
$S_2 = \cos A\cos B\cos C\sin^3\lambda$

$S_3 = \sin A \cos B \cos C \sin^2 \lambda \cos \lambda$
$S_4 = \sin A \cos B \sin C \sin \lambda \cos^2 \lambda$
$S_5 = \cos A \sin B \sin C \sin \lambda \cos^2 \lambda$
$S_6 = \cos A \sin B \cos C \sin^2 \lambda \cos \lambda$
$S_7 = \sin A \sin B \cos C \sin \lambda \cos^2 \lambda$
$S_8 = \sin A \sin B \sin C \cos^3 \lambda$

Applying the formulas of sine of the addition and the sine of the substraction to the three first factors that appear in the sum of the numerators IV, V, VI

$\sin(A-\lambda)\sin(B+\lambda)\sin(C+\lambda) =$
$(\sin A \cos\lambda - \cos A \sin\lambda)(\sin B \cos\lambda + \cos B \sin\lambda)(\sin C \cos\lambda + \cos C \sin\lambda) =$
$-S_1 - S_2 + S_3 + S_4 - S_5 - S_6 + S_7 + S_8$

$\sin(B-\lambda)\sin(A+\lambda)\sin(C+\lambda) =$
$(\sin B \cos\lambda - \cos B \sin\lambda)(\sin A \cos\lambda + \cos A \sin\lambda)(\sin C \cos\lambda + \cos C \sin\lambda) =$
$-S_1 - S_2 - S_3 - S_4 + S_5 + S_6 + S_7 + S_8$

$\sin(C-\lambda)\sin(A+\lambda)\sin(B+\lambda) =$
$(\sin C \cos\lambda - \cos C \sin\lambda)(\sin A \cos\lambda + \cos A \sin\lambda)(\sin B \cos\lambda + \cos B \sin\lambda) =$
$S_1 - S_2 - S_3 + S_4 + S_5 - S_6 - S_7 + S_8$

If we substitute these results, the sum of the numerators IV, V, VI, would appear like that

$2\big[\cos(2A-\lambda)(-S_1 - S_4 + S_6 + S_7) + \cos(2B-\lambda)(S_1 - S_3 + S_5 - S_7) + \cos(2C-\lambda)(S_3 + S_4 - S_5 - S_6)\big] =$

$2(S_7 - S_1)\big[\cos(2A-\lambda) - \cos(2B-\lambda)\big] + (S_5 - S_3)\big[\cos(2B-\lambda) - \cos(2C-\lambda)\big] + (S_6 - S_4)\big[\cos(2A-\lambda) - \cos(2C-\lambda)\big] \qquad \text{(VII)}$

and now

$$(S_7 - S_1) = (\sin\lambda\cos\lambda)(\underbrace{\sin A \sin B \cos C \cos\lambda}_{K_1} - \underbrace{\cos A \cos B \sin C \sin\lambda}_{K_2}) = (\sin\lambda\cos\lambda)(K_1 - K_2) \qquad \text{(VIII)}$$

$$(S_5 - S_3) = (\sin\lambda\cos\lambda)(\underbrace{\cos A \sin B \sin C \cos\lambda}_{K_3} - \underbrace{\sin A \cos B \cos C \sin\lambda}_{K_4}) = (\sin\lambda\cos\lambda)(K_3 - K_4) \qquad \text{(IX)}$$

$$(S_6 - S_4) = (\sin\lambda\cos\lambda)(\underbrace{\cos A \sin B \cos C \sin\lambda}_{K_5} - \underbrace{\sin A \cos B \sin C \cos\lambda}_{K_6}) = (\sin\lambda\cos\lambda)(K_5 - K_6) \qquad \text{(X)}$$

Using the formula

$\boxed{\cos\alpha - \cos\beta = -2\sin\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}}$ and verifying that $\boxed{\hat{A} + \hat{B} + \hat{C} = \pi\, rad}$

$\cos(2A-\lambda) - \cos(2B-\lambda) = -2\sin(A+B-\lambda)\sin(A-B) =$
$-2\sin\big[(\pi - C) - \lambda\big]\sin(A-B) = -2\big[\sin(\pi-C)\cos\lambda - \cos(\pi-C)\sin\lambda\big]\sin(A-B) =$
$-2(\sin C\cos\lambda + \cos C \sin\lambda)(\sin A \cos B - \cos A \sin B) =$

$$-2(-\underbrace{\cos A \sin B \sin C \cos\lambda}_{K_3} + \underbrace{\sin A \sin B \cos C \sin\lambda}_{K_4} - \underbrace{\cos A \sin B \cos C \sin\lambda}_{K_5} + \underbrace{\sin A \cos B \sin C \cos\lambda}_{K_6}) =$$

$-2(-K_3 + K_4 - K_5 + K_6) \qquad \text{(XI)}$

$\cos(2B-\lambda) - \cos(2C-\lambda) = -2\big[\sin(\pi - A) - \lambda\big]\sin(B-C) =$

$$-2(\underbrace{\sin A \sin B \cos C \cos\lambda}_{K_1} - \underbrace{\cos A \cos B \sin C \sin\lambda}_{K_2} + \underbrace{\cos A \sin B \cos C \sin\lambda}_{K_5} - \underbrace{\sin A \cos B \sin C \cos\lambda}_{K_6}) =$$

$-2(K_1 - K_2 + K_5 - K_6) \qquad \text{(XII)}$

$\cos(2A-\lambda) - \cos(2C-\lambda) = -2\big[\sin(\pi - B) - \lambda\big]\sin(A-C) =$

$$-2(\underbrace{\sin A \sin B \cos C \cos \lambda}_{K_1} - \underbrace{\cos A \cos B \sin C \sin \lambda}_{K_2} - \underbrace{\cos A \sin B \sin C \cos \lambda}_{K_3} + \underbrace{\sin A \cos B \cos C \sin \lambda}_{K_4}) =$$

$$-2(K_1 - K_2 - K_3 + K_4) \qquad \text{(XIII)}$$

We substitute in VII the expressions VIII, IX, X, XI, XII y XIII

$$(-4\sin\lambda\cos\lambda)\big[(K_1 - K_2)(-K_3 + K_4 - K_5 + K_6) + (K_3 - K_4)(K_1 - K_2 + K_5 - K_6) + (K_5 - K_6)(K_1 - K_2 - K_3 + K_4)\big] =$$

$$(-4\sin\lambda\cos\lambda)\big(K_1K_3 - K_1K_3 + K_1K_4 - K_1K_4 + K_1K_5 - K_1K_5 + K_1K_6 - K_1K_6 + K_2K_3 - K_2K_3 + K_2K_4 - K_2K_4 + K_2K_5 - K_2K_5 + K_2K_6 - K_2K_6 + K_3K_5 - K_3K_5 + K_3K_6 - K_3K_6 + K_4K_5 - K_4K_5 + K_4K_6 - K_4K_6) =$$

$$(-4\sin\lambda\cos\lambda)\,0 = 0$$