Applying the Menelaus theorem to ΔNII_a with transversal AA', we obtain

$$\frac{IF'}{F'N}\cdot\frac{NA'}{A'I_a}\cdot\frac{I_aA}{IA} = 1,$$

so

$$\frac{IF'}{F'N} = \frac{A'I_a}{NA'}\cdot\frac{IA}{I_aA}. \tag{4}$$

We also have $\Delta AZI \sim \Delta AZ_aI_a$, so using (1),

$$\frac{IA}{I_aA} = \frac{AZ}{AZ_a} = \frac{s-a}{s}. \tag{5}$$

Since $A'I_a = r_a$ and $NA' = q = R/2$, using (2) and (5) in (4) we obtain

$$\frac{IF'}{F'N} = \frac{2r_a(s-a)}{Rs} = \frac{2rs}{Rs} = \frac{2r}{R}. \tag{6}$$

The ratio in (6) defines a unique point F' of the segment NI. The same argument applied to BB' and CC' would lead to the same equation (6) for the points on NI through which they pass. It follows that all three lines AA', BB', CC' are concurrent at the point F' on NI defined by (6).

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; and the proposer.

The problem is also contained in Theorem 5 of Andrew P. Guinand's paper "Graves triads in the geometry of the triangle", Journal of Geometry 6/2 (1975) 131-142. *I thank the proposer for since pointing this out.*

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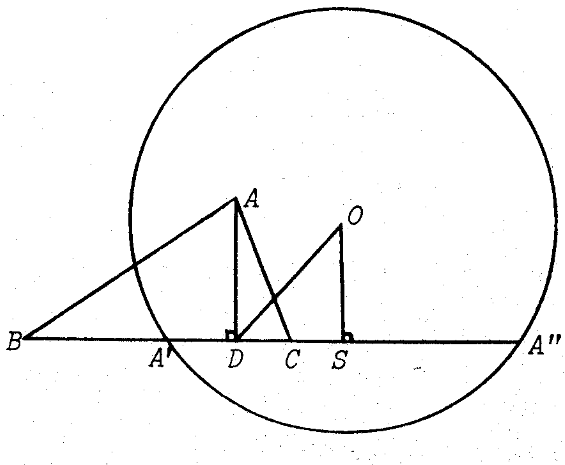
<u>1140</u>. [1986: 79] *Proposed by Jordi Dou, Barcelona, Spain.*

Given triangle ABC, construct a circle which cuts (extended) lines BC, CA, AB in pairs of points A' and A'', B' and B'', C' and C'' respectively such that angles $A'AA''$, $B'BB''$, $C'CC''$ are all right angles.

Solution by Dan Sokolowsky, Williamsburg, Virginia.

We first prove

(I) Let a circle K with center O and radius r meet side BC of ΔABC at A' and A''. Let D be the foot of the perpendicular from A to BC, and let $h_a = AD$, $x = OD$. Then $\angle A'AA'' = 90^\circ$ if and only if $r^2 = h_a^2 + x^2$.



To prove this, let S be the midpoint of $A'A''$, so $OS \perp A'A''$. Then

$$\begin{aligned}\angle A'AA'' = 90^\circ &\Leftrightarrow (A'S)^2 = (AS)^2 \\ &\Leftrightarrow (A'O)^2 - (OS)^2 = (AD)^2 + (DS)^2 \\ &\Leftrightarrow r^2 = (A'O)^2 = (AD)^2 + (DS)^2 + (OS)^2 = (AD)^2 + (OD)^2 \\ &\qquad = h_a^2 + x^2.\end{aligned}$$

To apply this, given a ΔABC let $h_a = AD$, $h_b = BE$, $h_c = CF$, and for any point O let $OD = x$, $OE = y$, $OF = z$. By (I), if a circle $K = O(r)$ solves the problem, we must have

$$r^2 = h_a^2 + x^2 \tag{1}$$

$$r^2 = h_b^2 + y^2 \tag{2}$$

$$r^2 = h_c^2 + z^2, \tag{3}$$

which imply

$$x^2 - y^2 = h_b^2 - h_a^2 \tag{4}$$

$$y^2 - z^2 = h_c^2 - h_b^2 \tag{5}$$

$$z^2 - x^2 = h_a^2 - h_c^2. \tag{6}$$

If $D \neq E$, the locus of points O satisfying (4) is well known to be a line, call it L_1, perpendicular to DE, and easily constructed. (For details see [1], [2].) Likewise if $E \neq F$ the points O satisfying (5) form a line L_2 perpendicular to EF, while if $D \neq F$ those satisfying (6) form a line L_3 perpendicular to DF.

We have two cases:

(i) ΔABC is not a right triangle.

Then D, E, F are distinct non-collinear points, and no two of L_1, L_2, L_3 are parallel. In fact they are concurrent as can be seen by noting that any two of the equations (4) - (6) imply the third, so that if, say, L_1 and L_2 meet at O, then O satisfies (4) and (5), hence (6), implying L_3 also passes through O. Thus O can be constructed as the intersection of any two of these three lines.

Since (4) - (6) hold at such a point O we have there

$$h_a^2 + x^2 = h_b^2 + y^2 = h_c^2 + z^2 \tag{7}$$

so if we construct r so as to satisfy (1) (which must necessarily be satisfied), then (2) and (3) are satisfied as well, by virtue of (7). By (I), the corresponding circle $K = O(r)$ solves our problem. Moreover, the argument shows that in this case O and r are unique, hence so is the solution.

(ii) ΔABC is a right triangle.

We can suppose $\angle C = 90^\circ$, in which case $h_a = b$, $h_b = a$, while D and E coincide (with C) so $x = y$ for any point O. Here the lines DF and EF coincide (with CF), so the corresponding lines L_2 and L_3 are parallel. We consider the following two subcases:

(a) $a \neq b$.

Hence $h_a \neq h_b$. Since $x = y$ for any point O, there exist no points satisfying (4), which implies that in this case there is no solution for our problem. Also implied is the fact that L_2 and L_3 are not coincident since any point they had in common would satisfy (4).

Note however that (5) holds for any point O on L_2, so that at any such point

$$h_b^2 + y^2 = h_c^2 + z^2.$$

Thus if at this point O we construct the corresponding r satisfying (2) it satisfies (3) as well, hence by (I), for the corresponding circle $K = O(r)$ we have

$$\angle B'BB'' = \angle C'CC'' = 90^\circ. \tag{8}$$

We can likewise argue that for any point O on L_3 there is a circle $K = O(r)$ such that

$$\angle A'AA'' = \angle C'CC'' = 90^\circ. \tag{9}$$

Since O is arbitrary in either case, there are infinitely many circles K satisfying either (8) or (9), but as the argument shows, there are none satisfying both in common.

(b) $a = b$.

Then $h_a = h_b$, and since $x = y$, any point O satisfying either (5) or (6) satisfies the other, implying that here L_2 and L_3 coincide, call them L. Then by the same argument as in the preceding case, for any point O on L we can show there is a circle $K = O(r)$ for which both (8) and (9) hold, hence is a solution. O being arbitrary, it follows that in this case there are infinitely many solutions. It is easy to show that the point C lies on L, hence that L is the line through C perpendicular to CF (equivalently, that L is parallel to AB).

References.

[1] R.A. Johnson, *Advanced Euclidean Geometry*, pp.31-33.

[2] H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited*, pp.30-34.

Also solved by NIELS BEJLEGAARD, Stavanger, Norway; and the proposer.

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