1140. [1986 : 79; 1987 : 232–235] *Proposed by Jordi Dou, Barcelona, Spain.*

Given triangle ABC, construct a circle which cuts (extended) lines BC, CA, AB in pairs of points A' and A'', B' and B'', C' and C'', respectively such that angles $A'AA''$, $B'BB''$, $C'CC''$ are all right angles.

Ricardo Barroso Campos has suggested that we publish Dou's lovely solution to celebrate his 91st birthday. We thank Professor Barroso for his English translation.

Editor's note.
Dou's solution follows from three basic results from projective geometry:

i. The lines through A define a projective involution I_1 on the points of BC by the rule that P' is the image of P when $\angle P'AP$ is a right angle. (Note that under I_1 the foot of the altitude from A to BC is interchanged with the point at infinity of BC.) Similarly I_2 is defined on CA by the perpendicular pairs of lines through B, and I_3 on AB by the perpendicular pairs of lines through C.

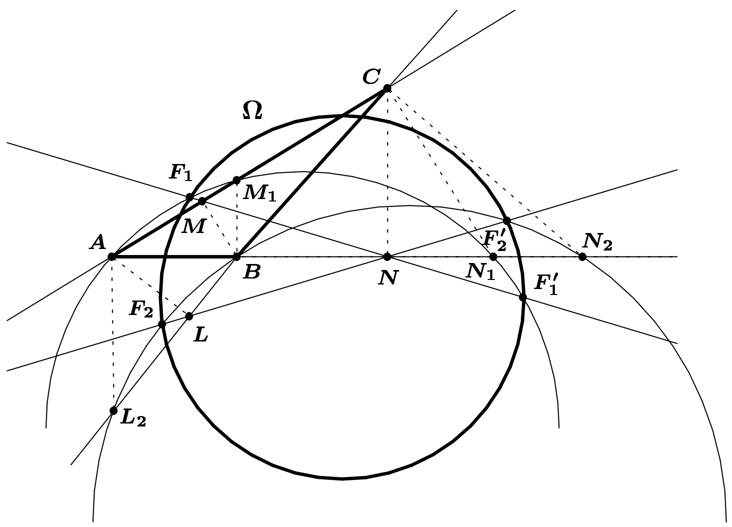
ii. For any four points W, X, Y, Z on a line there exists one and only one projective involution of that line that interchanges W with X and Y with Z.

iii. (The Desargues Involution Theorem) Each circle of the pencil of circles through two fixed points Q and Q' meets any line through neither Q nor Q' in pairs of points that are interchanged by a projective involution.

Solution by Jordi Dou, Barcelona, Spain.

First assume that $\triangle ABC$ is not a right triangle. Let L, M, N be the feet of the altitudes from A, B, C, respectively. Define

M_1	on	CA	so that	$\angle ABM_1$	$=$	$90°$,	
N_1	on	AB	so that	$\angle ACN_1$	$=$	$90°$,	
N_2	on	AB	so that	$\angle BCN_2$	$=$	$90°$,	and
L_2	on	BC	so that	$\angle BAL_2$	$=$	$90°$.	See figure below.



Let ϕ_1 be the pencil of circles through the points F_1 and F_1' where line MN intersects circle AM_1N_1, and let ϕ_2 be the pencil of circles through the points F_2 and F_2' where the line NL intersects circle BN_2L_2. By definition, the pencil ϕ_1 contains "circles" AM_1N_1 and $MN\infty$; the pencil ϕ_1 therefore induces involutions:

$$\text{on } AC \text{ that switches } A \text{ with } M_1 \quad \text{and} \quad M \text{ with } \infty\,,$$

and

$$\text{on } AB \text{ that switches } A \text{ with } N_1 \quad \text{and} \quad N \text{ with } \infty\,.$$

Note that these involutions agree with I_2 for two pairs of points on AC (because $\angle ABM_1 = \angle MB\infty = 90°$), and with I_3 on AB (because $\angle ACN_1 = \angle NC\infty = 90°$). Similarly, ϕ_2 (with circles BN_2L_2 and $NL\infty$) induces I_3 on AB ($\angle BCN_2 = \angle NC\infty = 90°$) and I_1 on BC ($\angle BAL_2 = \angle LA\infty = 90°$). Finally circle Ω through $F_1F_1'F_2$ must also pass through F_2' since Ω (being in the pencil ϕ_1) meets AB in a pair of I_3 so that Ω is also in ϕ_2. Consequently, Ω is a solution to the problem. Moreover, the pairs $\{A', A''\}$, $\{B', B''\}$, and $\{C', C''\}$, where Ω intersects the sides of $\triangle ABC$, always exist and are unique because I_1, I_2, I_3 are uniquely defined *elliptic* involutions, where "elliptic" means that no point equals its image under the involution.

When there is a right angle at C, the solution circle Ω becomes the extension of the altitude from C to the hypotenuse (using the convention, as in the extended Gaussian plane, that all lines intersect in a single point at infinity). When the right triangle is isosceles there will be further solution circles. See Sokolowsky's discussion [1987 : 235].