Problema 618. Sea ABC un triángulo rectángulo en A con catetos c (fijo) y b (variable), y con hipotenusa a variable. Definimos h_a, w_a, m_a la altura, bisectriz y mediana correspondientes a la hipotenusa a. Sean m_b y m_c las medianas correspondientes a los catetos b y c, respectivamente. Calcular los siguientes límites:

i) $$A=\lim_{b\to c}\frac{\sqrt{\frac{1}{b^2}+\frac{1}{c^2}}-\frac{1}{m_a}}{(b-c)^2}$$

ii) $$A=\lim_{b\to c}\frac{\sqrt{\frac{1}{b^2}+\frac{1}{c^2}}-\frac{1}{w_a}}{(b-c)^2}$$

iii) $$A=\lim_{b\to c}\frac{\sqrt{m_b^2+m_c^2}-\sqrt{5}h_a}{(b-c)^2}$$

iv) $$A=\lim_{b\to c}\frac{\sqrt{m_b^2+m_c^2}-\sqrt{5}w_a}{(b-c)^2}$$

Romero, J.B. (2011): Comunicación personal.

Solución de Bruno Salgueiro Fanego, Viveiro, Lugo.

$$h_a=\frac{bc}{a}=\frac{bc}{\sqrt{b^2+c^2}},\quad w_a=\frac{2bc\cos\left(\frac{A}{2}\right)}{b+c}=\frac{\sqrt{2}bc}{b+c},$$
$$m_a=\frac{\sqrt{2\left(b^2+c^2\right)-a^2}}{2}=\frac{\sqrt{2\left(b^2+c^2\right)-\left(b^2+c^2\right)}}{2}=\frac{\sqrt{b^2+c^2}}{2},\quad m_b=\frac{\sqrt{2\left(c^2+a^2\right)-b^2}}{2},$$
$$m_c=\frac{\sqrt{2\left(a^2+b^2\right)-c^2}}{2};$$

i) $$A=\lim_{b\to c}\frac{\sqrt{\frac{1}{b^2}+\frac{1}{c^2}}-\frac{1}{m_a}}{(b-c)^2}=\lim_{b\to c}\frac{\sqrt{\frac{b^2+c^2}{b^2c^2}}-\frac{2}{\sqrt{b^2+c^2}}}{(b-c)^2}=\lim_{b\to c}\frac{\frac{\sqrt{b^2+c^2}}{|bc|}-\frac{2}{\sqrt{b^2+c^2}}}{(b-c)^2}$$

$$= \lim_{b\to c}\frac{\dfrac{\left(\sqrt{b^2+c^2}\right)^2-2bc}{bc\sqrt{b^2+c^2}}}{(b-c)^2} = \lim_{b\to c}\frac{b^2+c^2-2bc}{bc\sqrt{b^2+c^2}\,(b-c)^2} = \lim_{b\to c}\frac{(b-c)^2}{bc\sqrt{b^2+c^2}\,(b-c)^2}$$

$$= \lim_{b\to c}\frac{1}{bc\sqrt{b^2+c^2}} = \frac{1}{c^2\sqrt{c^2+c^2}} = \frac{1}{c^2\sqrt{2}|c|} = \frac{\sqrt{2}}{2c^3}$$

ii) $$A = \lim_{b\to c}\frac{\sqrt{\dfrac{1}{b^2}+\dfrac{1}{c^2}}-\dfrac{1}{w_a}}{(b-c)^2} = \lim_{b\to c}\frac{\dfrac{\sqrt{b^2+c^2}}{bc}-\dfrac{b+c}{\sqrt{2}bc}}{(b-c)^2} = \lim_{b\to c}\frac{\sqrt{2\left(b^2+c^2\right)}-(b+c)}{\sqrt{2}bc(b-c)^2}$$

$$= \lim_{b\to c}\frac{\left[\sqrt{2\left(b^2+c^2\right)}-(b+c)\right]\left[\sqrt{2\left(b^2+c^2\right)}+(b+c)\right]}{\sqrt{2}bc(b-c)^2\left[\sqrt{2\left(b^2+c^2\right)}+(b+c)\right]} = \lim_{b\to c}\frac{\left[\sqrt{2\left(b^2+c^2\right)}\right]^2-(b+c)^2}{\sqrt{2}bc(b-c)^2\left[\sqrt{2\left(b^2+c^2\right)}+(b+c)\right]}$$

$$= \lim_{b\to c}\frac{2\left(b^2+c^2\right)-\left(b^2+c^2+2bc\right)}{\sqrt{2}bc(b-c)^2\left[\sqrt{2\left(b^2+c^2\right)}+(b+c)\right]} = \lim_{b\to c}\frac{(b-c)^2}{\sqrt{2}bc(b-c)^2\left[\sqrt{2\left(b^2+c^2\right)}+(b+c)\right]}$$

$$= \lim_{b\to c}\frac{1}{\sqrt{2}bc\left[\sqrt{2\left(b^2+c^2\right)}+(b+c)\right]} = \frac{1}{\sqrt{2}c^2\left[\sqrt{2\left(c^2+c^2\right)}+2c\right]} = \frac{1}{\sqrt{2}c^2(2c+2c)} = \frac{\sqrt{2}}{8c^3}$$

iii) $$A = \lim_{b\to c}\frac{\sqrt{m_b^2+m_c^2}-\sqrt{5}h_a}{(b-c)^2} = \lim_{b\to c}\frac{\sqrt{\dfrac{2\left(c^2+a^2\right)-b^2}{4}+\dfrac{2\left(a^2+b^2\right)-c^2}{4}}-\dfrac{\sqrt{5}bc}{\sqrt{b^2+c^2}}}{(b-c)^2}$$

$$= \lim_{b\to c}\frac{\sqrt{\dfrac{4a^2+b^2+c^2}{4}}-\dfrac{\sqrt{5}bc}{\sqrt{b^2+c^2}}}{(b-c)^2} = \lim_{b\to c}\frac{\dfrac{\sqrt{5\left(b^2+c^2\right)}}{2}-\dfrac{\sqrt{5}bc}{\sqrt{b^2+c^2}}}{(b-c)^2} = \lim_{b\to c}\frac{\sqrt{5}\left(b^2+c^2-2bc\right)}{2(b-c)^2\sqrt{b^2+c^2}}$$

$$= \lim_{b\to c}\frac{\sqrt{5}(b-c)^2}{2(b-c)^2\sqrt{b^2+c^2}} = \lim_{b\to c}\frac{\sqrt{5}}{2\sqrt{b^2+c^2}} = \frac{\sqrt{5}}{2\sqrt{c^2+c^2}} = \frac{\sqrt{5}}{2\sqrt{2c^2}} = \frac{\sqrt{10}}{4c}$$

iv) $$A=\lim_{b\to c}\frac{\sqrt{m_b^2+m_c^2}-\sqrt{5}w_a}{(b-c)^2}=\lim_{b\to c}\frac{\frac{\sqrt{5(b^2+c^2)}}{2}-\frac{\sqrt{5}\sqrt{2}bc}{b+c}}{(b-c)^2}=\lim_{b\to c}\frac{\sqrt{5}\left[\sqrt{b^2+c^2}(b+c)-2\sqrt{2}bc\right]}{2(b+c)(b-c)^2}$$

$$=\lim_{b\to c}\frac{\sqrt{5}\left[\sqrt{b^2+c^2}(b+c)-2\sqrt{2}bc\right]\left[\sqrt{b^2+c^2}(b+c)+2\sqrt{2}bc\right]}{2(b+c)(b-c)^2\left[\sqrt{b^2+c^2}(b+c)+2\sqrt{2}bc\right]}$$

$$=\lim_{b\to c}\frac{\sqrt{5}\left[\left(\sqrt{b^2+c^2}\right)^2(b+c)^2-\left(2\sqrt{2}bc\right)^2\right]}{2(b+c)(b-c)^2\left[\sqrt{b^2+c^2}(b+c)+2\sqrt{2}bc\right]}=\lim_{b\to c}\frac{\sqrt{5}\left[\left(b^2+c^2-2bc+2bc\right)(b+c)^2-8b^2c^2\right]}{2(b+c)(b-c)^2\left[\sqrt{b^2+c^2}(b+c)+2\sqrt{2}bc\right]}$$

$$=\lim_{b\to c}\frac{\sqrt{5}\left[(b-c)^2(b+c)^2+2bc(b+c)^2-8b^2c^2\right]}{2(b+c)(b-c)^2\left[\sqrt{b^2+c^2}(b+c)+2\sqrt{2}bc\right]}=\lim_{b\to c}\frac{\sqrt{5}\left[(b-c)^2(b+c)^2+2bc\left(b^2+c^2+2bc-4bc\right)\right]}{2(b+c)(b-c)^2\left[\sqrt{b^2+c^2}(b+c)+2\sqrt{2}bc\right]}$$

$$=\lim_{b\to c}\frac{\sqrt{5}\left[(b-c)^2(b+c)^2+2bc(b-c)^2\right]}{2(b+c)(b-c)^2\left[\sqrt{b^2+c^2}(b+c)+2\sqrt{2}bc\right]}=\lim_{b\to c}\frac{\sqrt{5}(b-c)^2\left[(b+c)^2+2bc\right]}{2(b+c)(b-c)^2\left[\sqrt{b^2+c^2}(b+c)+2\sqrt{2}bc\right]}$$

$$=\lim_{b\to c}\frac{\sqrt{5}\left[(b+c)^2+2bc\right]}{2(b+c)\left[\sqrt{b^2+c^2}(b+c)+2\sqrt{2}bc\right]}=\frac{\sqrt{5}\left[(c+c)^2+2c^2\right]}{2(c+c)\left[\sqrt{c^2+c^2}(c+c)+2\sqrt{2}c^2\right]}$$

$$=\frac{6\sqrt{5}c^2}{4c\left(2\sqrt{2}c^2+2\sqrt{2}c^2\right)}=\frac{3\sqrt{5}c^2}{4c2\sqrt{2}c^2}=\frac{3\sqrt{10}}{16c}$$