Problema 665, propuesto por Juan Bosco Romero Márquez, Valladolid:

Si A, B y C son los ángulos de un triángulo, siendo A el mayor, probar que:

$$\prod_{\text{cíclico}} \cos\frac{A}{2} - \prod_{\text{cíclico}} \operatorname{sen}\frac{A}{2} \underset{\leq}{\geq} \frac{1}{2} \Leftrightarrow \mathrm{A} \underset{\geq}{\leq} 90^\circ .$$

Solución 2 de Bruno Salgueiro Fanego, Viveiro, Lugo:

Sean $\alpha = \dfrac{A}{2}$, $\beta = \dfrac{B}{2}$ y $\gamma = \dfrac{C}{2}$. Al ser $\alpha + \beta + \gamma = \dfrac{1}{2}(A+B+C) = 90^\circ$, resulta que

$$\begin{aligned}
&0+1i=\operatorname{sen}\gamma\cos\gamma - \cos\gamma\operatorname{sen}\gamma + i\left(\operatorname{sen}^2\gamma + \cos^2\gamma\right) = \left(\operatorname{sen}\gamma + i\cos\gamma\right)\left(\cos\gamma + i\operatorname{sen}\gamma\right)\\
&=\left(\cos\left(\frac{\pi}{2}-\gamma\right) + i\operatorname{sen}\left(\frac{\pi}{2}-\gamma\right)\right)\left(\cos\gamma + i\operatorname{sen}\gamma\right) = \left(\cos(\alpha+\beta) + i\operatorname{sen}(\alpha+\beta)\right)\left(\cos\gamma + i\operatorname{sen}\gamma\right)\\
&=\left(\cos\alpha\cos\beta - \operatorname{sen}\alpha\operatorname{sen}\beta + i\left(\cos\alpha\operatorname{sen}\beta + \operatorname{sen}\alpha\cos\beta\right)\right)\left(\cos\gamma + i\operatorname{sen}\gamma\right)\\
&=\cos\alpha\cos\beta\cos\gamma - \operatorname{sen}\alpha\operatorname{sen}\beta\cos\gamma - \cos\alpha\operatorname{sen}\beta\operatorname{sen}\gamma - \operatorname{sen}\alpha\cos\beta\operatorname{sen}\gamma + i\left(\cos\alpha\cos\beta\operatorname{sen}\gamma\right.\\
&\left.-\operatorname{sen}\alpha\operatorname{sen}\beta\operatorname{sen}\gamma + \cos\alpha\operatorname{sen}\beta\cos\gamma + \operatorname{sen}\alpha\cos\beta\cos\gamma\right)\\
&\Rightarrow \begin{cases} 0 = \cos\alpha\cos\beta\cos\gamma - \operatorname{sen}\alpha\operatorname{sen}\beta\cos\gamma - \cos\alpha\operatorname{sen}\beta\operatorname{sen}\gamma - \operatorname{sen}\alpha\cos\beta\operatorname{sen}\gamma \\ 1 = \cos\alpha\cos\beta\operatorname{sen}\gamma - \operatorname{sen}\alpha\operatorname{sen}\beta\operatorname{sen}\gamma + \cos\alpha\operatorname{sen}\beta\cos\gamma + \operatorname{sen}\alpha\cos\beta\cos\gamma \end{cases}\\
&\Rightarrow \begin{cases} \operatorname{sen}\alpha\operatorname{sen}\beta\cos\gamma + \cos\alpha\operatorname{sen}\beta\operatorname{sen}\gamma + \operatorname{sen}\alpha\cos\beta\operatorname{sen}\gamma = \cos\alpha\cos\beta\cos\gamma. \\ 1 + \operatorname{sen}\alpha\operatorname{sen}\beta\operatorname{sen}\gamma = \cos\alpha\cos\beta\operatorname{sen}\gamma + \cos\alpha\operatorname{sen}\beta\cos\gamma + \operatorname{sen}\alpha\cos\beta\cos\gamma. \end{cases}
\end{aligned}$$

De estas dos igualdades, se deduce que

$$\begin{aligned}
&\left(\cos\alpha - \operatorname{sen}\alpha\right)\left(\cos\beta - \operatorname{sen}\beta\right)\left(\cos\gamma - \operatorname{sen}\gamma\right) = \left(\cos\alpha\cos\beta + \operatorname{sen}\alpha\operatorname{sen}\beta - \operatorname{sen}\alpha\cos\beta - \cos\alpha\operatorname{sen}\beta\right)\left(\cos\gamma - \operatorname{sen}\gamma\right)\\
&= \cos\alpha\cos\beta\cos\gamma + \operatorname{sen}\alpha\operatorname{sen}\beta\cos\gamma + \operatorname{sen}\alpha\cos\beta\operatorname{sen}\gamma + \cos\alpha\operatorname{sen}\beta\operatorname{sen}\gamma\\
&-\operatorname{sen}\alpha\cos\beta\cos\gamma - \cos\alpha\operatorname{sen}\beta\cos\gamma - \cos\alpha\cos\beta\operatorname{sen}\gamma - \operatorname{sen}\alpha\operatorname{sen}\beta\operatorname{sen}\gamma = \cos\alpha\cos\beta\cos\gamma + \cos\alpha\cos\beta\cos\gamma\\
&-1 - \operatorname{sen}\alpha\operatorname{sen}\beta\operatorname{sen}\gamma - \operatorname{sen}\alpha\operatorname{sen}\beta\operatorname{sen}\gamma = 2\left(\cos\alpha\cos\beta\cos\gamma - \operatorname{sen}\alpha\operatorname{sen}\beta\operatorname{sen}\gamma\right) - 1 = 2\left(\prod_{\text{cíclico}}\cos\frac{A}{2} - \prod_{\text{cíclico}}\operatorname{sen}\frac{A}{2}\right) - 1\\
&\Rightarrow \prod_{\text{cíclico}}\cos\frac{A}{2} - \prod_{\text{cíclico}}\operatorname{sen}\frac{A}{2} = \frac{1}{2}\left(\cos\alpha - \operatorname{sen}\alpha\right)\left(\cos\beta - \operatorname{sen}\beta\right)\left(\cos\gamma - \operatorname{sen}\gamma\right) + \frac{1}{2}
\end{aligned}$$

luego $\displaystyle\prod_{\text{cíclico}}\cos\frac{A}{2} - \prod_{\text{cíclico}}\operatorname{sen}\frac{A}{2} \underset{\leq}{\geq} \frac{1}{2} \Leftrightarrow \frac{1}{2}\left(\cos\alpha - \operatorname{sen}\alpha\right)\left(\cos\beta - \operatorname{sen}\beta\right)\left(\cos\gamma - \operatorname{sen}\gamma\right) + \frac{1}{2} \underset{\leq}{\geq} \frac{1}{2}$

$$\Leftrightarrow \left(\cos\alpha - \operatorname{sen}\alpha\right)\left(\cos\beta - \operatorname{sen}\beta\right)\left(\cos\gamma - \operatorname{sen}\gamma\right) \underset{\leq}{\geq} 0 .$$

Distinguimos los tres casos posibles:

1) El triángulo es acutángulo $\Leftrightarrow A < 90^\circ \Leftrightarrow A$, B y $C < 90^\circ \Leftrightarrow \alpha$, β y $\gamma < 45^\circ \Leftrightarrow \cos\alpha > \operatorname{sen}\alpha$, $\cos\beta > \operatorname{sen}\beta$ y $\cos\gamma > \operatorname{sen}\gamma \Leftrightarrow \cos\alpha - \operatorname{sen}\alpha > 0$, $\cos\beta - \operatorname{sen}\beta > 0$ y $\cos\gamma - \operatorname{sen}\gamma > 0$

$$\Leftrightarrow (\cos\alpha - \operatorname{sen}\alpha)(\cos\beta - \operatorname{sen}\beta)(\cos\gamma - \operatorname{sen}\gamma) > 0 \Leftrightarrow \prod_{\text{cíclico}} \cos\frac{A}{2} - \prod_{\text{cíclico}} \operatorname{sen}\frac{A}{2} > \frac{1}{2}.$$

2) El triángulo es rectángulo $\Leftrightarrow A = 90^\circ \Leftrightarrow A = 90^\circ$, B y $C < 90^\circ \Leftrightarrow \alpha = 45^\circ$, β y $\gamma < 45^\circ$

$$\Leftrightarrow \cos\alpha = \operatorname{sen}\alpha \Leftrightarrow (\cos\alpha - \operatorname{sen}\alpha)(\cos\beta - \operatorname{sen}\beta)(\cos\gamma - \operatorname{sen}\gamma) = 0 \Leftrightarrow \prod_{\text{cíclico}} \cos\frac{A}{2} - \prod_{\text{cíclico}} \operatorname{sen}\frac{A}{2} = \frac{1}{2}.$$

3) El triángulo es obtusángulo $\Leftrightarrow 90^\circ < A < 180^\circ \Leftrightarrow 90^\circ < A < 180^\circ$, B y $C < 90^\circ$

$\Leftrightarrow 45^\circ < \alpha < 90^\circ$, β y $\gamma < 45^\circ \Leftrightarrow \cos\alpha < \operatorname{sen}\alpha$, $\cos\beta > \operatorname{sen}\beta$ y $\cos\gamma > \operatorname{sen}\gamma \Leftrightarrow \cos\alpha - \operatorname{sen}\alpha < 0$, $\cos\beta - \operatorname{sen}\beta > 0$ y $\cos\gamma - \operatorname{sen}\gamma > 0 \Leftrightarrow (\cos\alpha - \operatorname{sen}\alpha)(\cos\beta - \operatorname{sen}\beta)(\cos\gamma - \operatorname{sen}\gamma) < 0$

$$\Leftrightarrow \prod_{\text{cíclico}} \cos\frac{A}{2} - \prod_{\text{cíclico}} \operatorname{sen}\frac{A}{2} < \frac{1}{2}.$$