EXPLORING THE SINORTHIAC TRIANGLES

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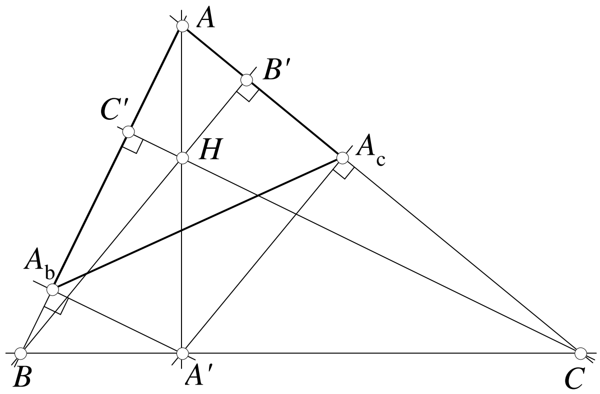
ON THE OCASSION OF THE 700TH *TRIANGULOSCABRI* PROBLEM

1. Introduction

We take as startpoint the information given in The Triangles Web website (TTW). Then we make some conjectures and use *Mathematica* to investigate about them. We use the notebook `Baricentricas.nb` to work with barycentric coordinates.

Definition 1. *The sinorthiac triangles of ABC have as vertices a vertex of ABC and the projections of the foot of its altitude on the two other sides.*

The following figure shows a triangle ABC, their three altitudes AA', BB', CC' and the sinorthiac triangle AA_bA_c corresponding to vertex A:



The following results are known:

Theorem 1 (r850)**.** *The Euler lines of the sinorthiac triangles AA_bA_c, BB_cB_a, CC_aC_b concur.*

Theorem 2 (r853)**.** *The medians of the sinorthiac triangles AA_bA_c, BB_cB_a, CC_aC_b coming from A, B, C concur at the symmedian K.*

Theorem 3 (r854)**.** *The altitudes of the sinorthiac triangles AA_bA_c, BB_cB_a, CC_aC_b coming from A, B, C concur at the circumcenter O.*

I dedicate this paper to the memory of my friends Juan Bosco Romero, José María Pedret and Manuel Prieto that passed away last year.

Remark. The intersection point of Euler lines in Theorem 1 is the point X_{974} in ETC.

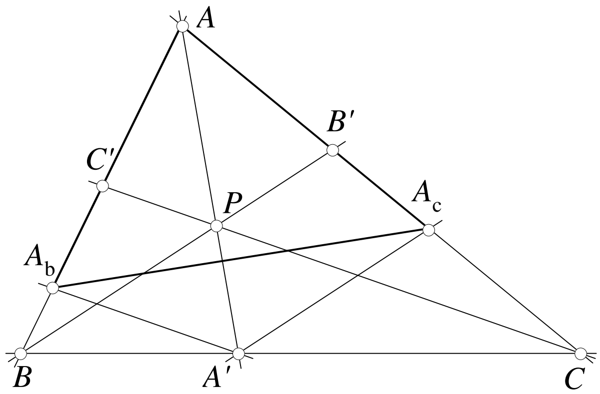
2. Generalization

Since the orthic triangle $A'B'C'$ of ABC is both the cevian and the pedal triangle of the orthocenter, we can easily generalize the sinorthiac triangles for a general point P in several ways.

2.1. **Pure cevian generalization.**

In this case we consider $A'B'C'$ as the cevian triangle of a point P and we draw parallels at A', B', C' to the other cevians.

Definition 2. *Let ABC be a triangle and P a point with cevian triangle $A'B'C'$. The parallel to BP, CP at A' intersect AC,AB at A_c, A_b respectively.*



Theorem 4. *The triangle AA_bA_c is isosceles with $AA_b = AA_c$ if and only if P lies on the circumhyperbola with the midpoint of BC as center.*

Proof. If $P = (x : y : z)$, we have $A_b = (xz : (x + y + z)y : 0)$ and $A_c = (xy : 0 : (x + y + z)z)$. The midpoint of A_bA_c is the point

$$N = (x(y^2 + z^2 + x(y + z)) : y(x + z)(x + y + z) : z(x + y)(x + y + z)),$$

and we can see that that this point lies on the bisector $AI : cy - bz = 0$ if and only if P satisfies

$$(x + y + z)(bxz + byz - cxy - cyz) = 0.$$

We can check that the circumconic $\mathcal{C}_a : bxz + byz - cxy - cyz = 0$ is invariant under the transformation $(x : y : z) \to (-x : x + z : x + y)$, that is the reflection on the midpoint of BC.

□

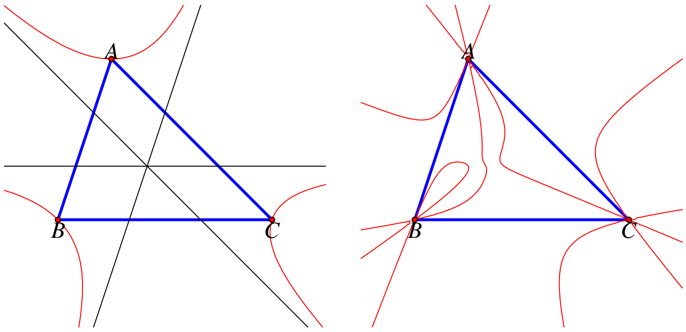
Remark. The circumconic $\mathcal{C}_a$ goes through the Gergonne point.

Corolary 1. *If P is the Gergonne point, then the three triangles AA_bA_c, BB_cB_a and CC_aC_b are isosceles.*

Corolary 2. *If P is the Gergonne point, then the Euler lines of the triangles AA_bA_c, BB_cB_a and CC_aC_b concur at the incenter of the triangle ABC.*

When P is the centroid, the three triangles AA_bA_c, BB_cB_a and CC_aC_b are the same: the triangle ABC. Thus, we have three coincident Euler lines in this case.

There are infinitely many cases when the Euler lines are concurrent. The locus is composed of the Tucker+ cubic (K016) and a sextic through G, H and the Gergonne point.



K016 is the only Tucker cubic with three concurring asymptotes and has equation

$$\sum_{\text{cyclic}} x(y^2+z^2)=0.$$

It may be instructive to make a drawing of the three concurrent Euler line for some particular point on $K016$.

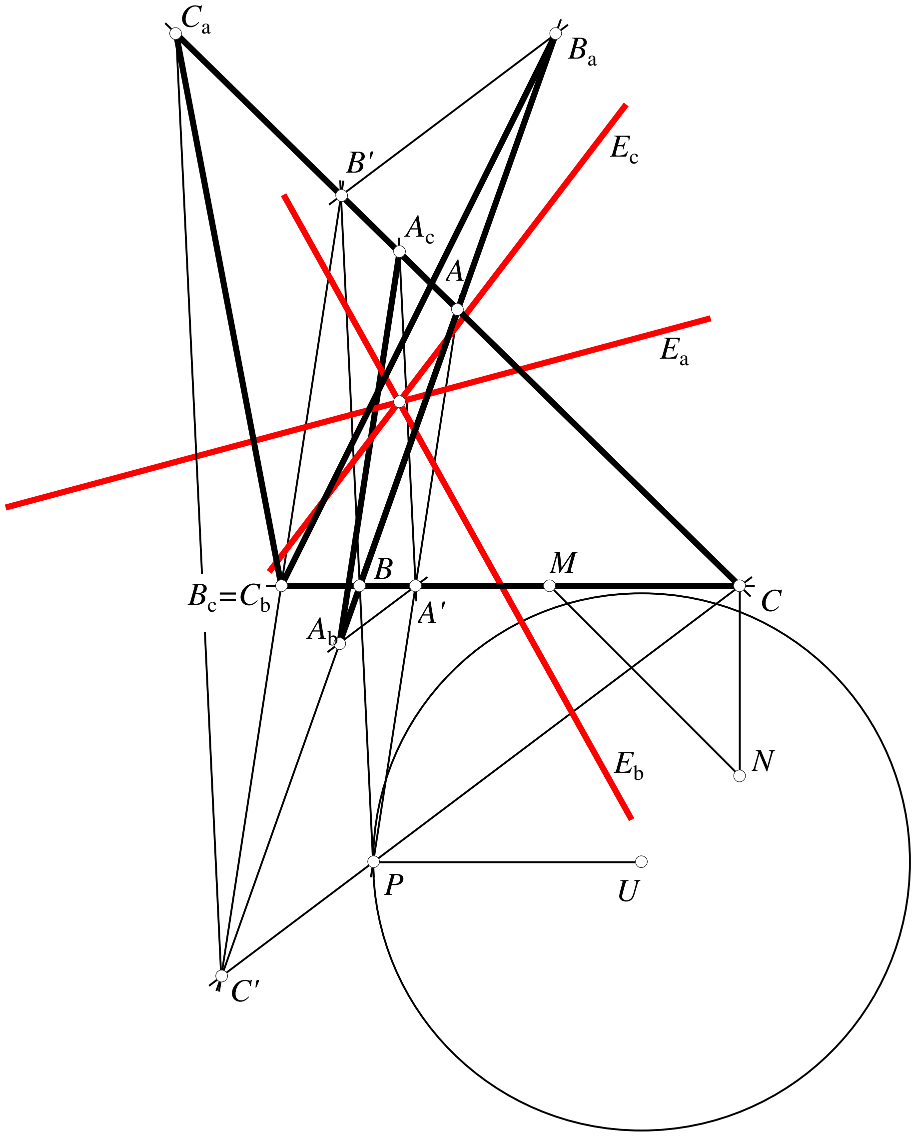
First of all, we look for a simple point on $K016$. For example, let $M=(0:1:1)$ be the midpoint of BC, and $U=(-1:1:1)$ the reflection of A on M. We will calculate the intersection of $K016$ and the line u through U parallel to BC.

The line u has equation $2x+y+z=0$ and this line intersects $K016$ at the points $P=(-2:2+\sqrt{2}:2-\sqrt{2})$,$P'=(-2:2-\sqrt{2}:2+\sqrt{2})$.

The midpoint of these points is U and the distance between them equals to $\sqrt{2}a$. This gives an easy construction.

Remark. We can observe in the figure the fact that $B_c=C_b$. This happens whenever for all points on the line u.

Remark. The line u seems to intersect the cubic at two points only. They also share the infinite point of BC.



Theorem 5. *The locus of P such that the altitudes of the triangles AA_bA_c, BB_cB_a, CC_aC_b coming from A, B, C are concurrent is the Lucas cubic. The intersection points lie on the Darboux cubic.*

The following table shows the intersection point Q of these three altitudes for some P lying on the Lucas cubic.

P	X_2	X_4	X_7	X_8	X_{20}	X_{69}
Q	X_4	X_3	X_1	X_{84}	X_{3346}	X_{64}

Theorem 6. *For any point P, the medians of the triangles AA_bA_c, BB_cB_a, CC_aC_b coming from A, B, C are concurrent at the crosspoint of P and G.*

Remark. If $P = (x : y : z)$, then the crosspoint of P and G has coordinates $(x(y+z) : y(z+x) : z(x+y))$ and it is the pole of the line PG with respect the conic $ABCPG$.

Corolary 3. *For any point P, the symmedians of the triangles AA_bA_c, BB_cB_a, CC_aC_b coming from A, B, C are concurrent at the isogonal conjugate of the crosspoint of P and G.*

Theorem 7. *For any point P, the six points $A_b, A_c, B_c, B_a, C_a, C_b$ lie on a conic. This conic is a circle if and only if P is the orthocenter.*

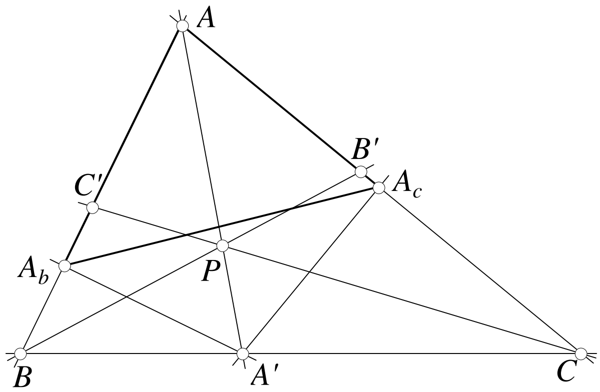
When P is the orthocenter, the circle through the six points is known as the *Taylor circle*. In the general case, for $P = (u : v : w)$ the conic has equation

$$\sum_{\text{cyclic}} v^2w^2(u+v+w)x^2 - uvw((u+v)^2 + (u+w)^2 + 2vw)yz = 0.$$

Its discriminant equals to $-uvw(u+v)^2(u+w)^2(v+w)^2(u+v+w)$, therefore we will have the same type of conic in each region determined by the sidelines of ABC and the sidelines of the anticomplementary triangle. In fact, the lines that separate regions with a different type of conic are the sidelines of ABC.

2.2. **Cevian triangle with projections.** We generalize the first definition in other way:

Definition 3. *Let ABC be a triangle and P a point with cevian triangle $A'B'C'$. Let A_b, A_c the orthogonal projections of A' on sides AB, AC respectively.*



In this case we get the following results:

Theorem 8. *The locus of P such that the medians of the triangles AA_bA_c, BB_cB_a, CC_aC_b coming from A, B, C are concurrent is the Darboux cubic. The intersection points lie on the Thomson cubic.*

The following table shows the intersection point Q of these three medians for some P lying on the Darboux cubic.

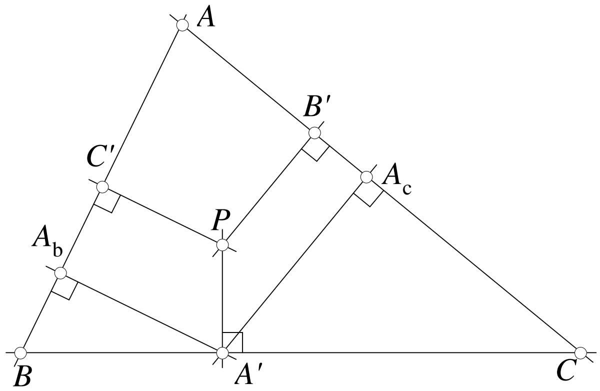
P	X_1	X_3	X_4	X_{20}	X_{40}	X_{64}	X_{84}
Q	X_1	X_2	X_6	X_3	X_9	X_4	X_{57}

Theorem 9. *For any point P, the altitudes of the triangles AA_bA_c, BB_cB_a, CC_aC_b coming from A, B, C are concurrent at the isogonal conjugate of P.*

Theorem 10. *The six points $A_b, A_c, B_c, B_a, C_a, C_b$ lie on a conic if and only if P lies on the Darboux cubic.*

2.3. **Pedal triangle with projections.** Now, we consider $A'B'C'$ as the pedal triangle of P:

Definition 4. *Let ABC be a triangle and P a point with pedal triangle $A'B'C'$. Let A_b, A_c the orthogonal projections of A' on sides AB, AC respectively.*



We find

Theorem 11. *The locus of P such that the altitudes of the triangles AA_bA_c, BB_cB_a, CC_aC_b coming from A, B, C are concurrent is the Darboux cubic. The intersection points lie on the $K172$ cubic, the isogonal conjugate of the Lucas cubic.*

The following table shows several points P on the Darboux cubic, the corresponding intersection point of the altitudes on $K172$ and its isogonal conjugate Q^* on the Lucas cubic.

P	X_1	X_3	X_4	X_{20}	X_{40}	X_{64}	X_{84}
Q	X_{55}	X_6	X_3	X_{25}	X_{56}	X_{154}	X_{198}
Q^*	X_7	X_2	X_4	X_{69}	X_8	X_{253}	X_{189}

3. Conclusion

The reader is invited to investigate other generalizations and to prove synthetically the theorems given here without proof.

References

[1] Castellsaguer, Quim. *The Triangles Web.*
`http://www.xtec.cat/~qcastell/ttw/`

[2] Gibert, Bernard. *Curves in the Triangle Plane.*
`http://bernard.gibert.pagesperso-orange.fr/index.html`

[3] Romero Marquez, Juan Bosco and García Capitán, Francisco Javier. *Planteando y resolviendo problemas de Geometría con Mathematica.*
`http://dmle.cindoc.csic.es/revistas/detalle.php?numero=6625`