

Problem 2

The point P lies inside an equilateral triangle and its distances to the three vertices are 3, 4, 5. Find the area of the triangle.

Solution. Label the vertices of the equilateral triangle A, B, C such that $PA = 3, PB = 4, PC = 5$. Rotate triangle BPA 60 degrees around A so that the edges AB and AC coincide. Let X be the image of point P . Observe that $PX = 3, CX = 4$, and $CP = 5$; thus, $PX \perp CX$. Also, APX becomes an equilateral triangle of side 3. Now, applying the law of cosines to triangle CXA , we find

$$AC = \sqrt{3^2 + 4^2 - 2 \cdot 3 \cdot 4 \cos 150^\circ} = \sqrt{25 + 12\sqrt{3}},$$

and the area of the triangle is $25\sqrt{3}/4 + 9$.