1054. (a*) Show how to construct triangle ABC by straightedge and compass, given side a, the median m_a to side a, and the angle bisector t_a to side a.

(b*) Show how to construct triangle ABC by straightedge and compass, given angle A, m_a, and t_a. [*Jerome C. Cherry, Santa Maria, California.*]

Solution: (a) We shall give an algebraic solution. Place the given side a on the x-axis of a rectangular Cartesian frame of reference, with the midpoint of the side at the origin. For convenience we choose our unit of distance equal to $a/2$ and designate m_a and t_a more simply by m and t, respectively. Let (x,y) denote the coordinates of vertex A, and $(-z,0)$ those of the foot of the angle bisector t. Then, utilizing the distance formula of analytic geometry and the fact that the angle bisector divides side a into parts proportional to the other two sides of the triangle, we have

$$x^2+y^2=m^2, \tag{1}$$

$$(x+z)^2+y^2=t^2, \tag{2}$$

$$\frac{(x+1)^2+y^2}{(x-1)^2+y^2}=\frac{(1-z)^2}{(1+z)^2}. \tag{3}$$

Setting $y^2=m^2-x^2$, from (1), in (2) and (3), and simplifying, we obtain

$$2xz+z^2+m^2=t^2, \tag{4}$$

$$\frac{1+2x+m^2}{1-2x+m^2}=\frac{(1-z)^2}{(1+z)^2}. \tag{5}$$

Eliminating x from (4) and (5) we obtain, after simplifying,

$$z^4-(t^2+m^2+1)z^2+(m^2-t^2)=0, \tag{6}$$

a quadratic in z^2. By standard constructions, we can construct segments of lengths t^2+m^2+1 and $(m^2-t^2)^{1/2}$. Again, by standard constructions we can construct a segment of length z^2, and then one of length z. Once z is found, the sought triangle is easily constructed.

Note. The devotee of the game of Euclidean constructions is not really interested in the actual mechanical construction of the sought triangle, but merely in the assurance that the construction is possible. To use a phrase of Jacob Steiner, the devotee performs his construction "simply by means of the tongue," rather than with actual instruments on paper. As soon as equation (6) is achieved, and recognized as a quadratic in z^2 with suitably constructible coefficients, the problem is finished. From this point of view, the problem is, in reality, essentially a pillow problem.

(b) Again we shall give an algebraic solution. Place vertex A at the origin of a rectangular Cartesian frame of reference, with t_a lying along the positive x-axis. To simplify the notation, designate t_a and m_a more simply by t and m. Let k, $-k$, s denote the slopes of the sides of angle A and of an arbitrary line L through the point $(t,0)$, and let (x_m,y_m) denote the coordinates of the midpoint of the segment cut off on line L by the sides of angle A. The equation of L and of the degenerate conic made up of the sides of angle A are, respectively,

$$y=s(x-t) \quad \text{and} \quad y^2=k^2x^2.$$

Eliminating y we obtain

$$x^2(s^2-k^2)-2s^2tx+s^2t^2=0. \tag{7}$$

Designating the roots of quadratic (7) by x_1 and x_2, we find

$$x_m = \frac{x_1 + x_2}{2} = \frac{s^2 t}{s^2 - k^2} \tag{8}$$

whence

$$y_m = \frac{sk^2 t}{s^2 - k^2}. \tag{9}$$

Eliminating s from (8) and (9) we obtain

$$y_m^2 = k^2 x_m^2 - k^2 t x_m. \tag{10}$$

But we also have

$$x_m^2 + y_m^2 = m^2. \tag{11}$$

Eliminating y_m from (10) and (11) we obtain

$$x_m^2(k^2+1) - k^2 t x_m - m^2 = 0. \tag{12}$$

But, since $k = \tan(A/2)$, this reduces to

$$x_m^2 - t(\sin^2(A/2)x_m - m^2(\cos^2(A/2)) = 0. \tag{13}$$

Simple constructions yield segments of lengths $t(\sin^2(A/2))$ and $m(\cos(A/2))$. A standard construction then yields x_m. Once x_m is found, the sought triangle is easily constructed.

Note. As in the former problem, the devotee quits the game as soon as equation (13) is attained. To find the point (x_m, y_m) we seek an intersection of a hyperbola and a circle. In general, to find an intersection of a conic and a circle is beyond the Euclidean tools. In our case, however, the circle is specially placed with respect to the hyperbola—its center lies at a vertex of the hyperbola.

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