Problem 3 clasa a IX-a

„Grigore Moisil" contest, Satu-Mare, 2009
România

Let six distinct points on a circle of center O.

a) Consider the straight line which joins the orthocenter of the triangle defined by three of these points with the orthocenter of the triangle defined by the other three points. Prove that all these straight lines are concurrent at a point X.

b). Consider the straight line which joins the orthocenter of the triangle defined by three of these points with the centroid of the triangle defined by the other three points. Prove that all these straight lines are concurrent at a point Y.

c). Prove that O, X and Y are collinear and Y is the midpoint of [OX].

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(full text).

Note 1. Grigore Moisil was a great Romanian mathematician (1906 – 1973).

Note 2 The point b) is the problem 705 proposed by Gàbor Hollò (2014)

Solution a) Let A_i, $i=\overline{1,6}$, six distinct points on the circle of center O, and let H_{123} and H_{456} the orthocenters of the triangles $A_1A_2A_3$ and $A_4A_5A_6$.

The straight line $H_{123}H_{456}$ has the vectorial ecuation:
$\overrightarrow{OM} = (1-t)\overrightarrow{OH_{123}} + t\cdot\overrightarrow{OH_{456}}$, where M is a variable point on $H_{123}H_{455}$ and "t" is a real number.

But $\overrightarrow{OH_{123}} = \overrightarrow{OA_1} + \overrightarrow{OA_2} + \overrightarrow{OA_3}$ and
$\overrightarrow{OH_{456}} = \overrightarrow{OA_4} + \overrightarrow{OA_5} + \overrightarrow{OA_6}$ (Sylvester)

Hence $\overrightarrow{OM} = (1-t)(\overrightarrow{OA_1} + \overrightarrow{OA_2} + \overrightarrow{OA_3}) + t\cdot(\overrightarrow{OA_4} + \overrightarrow{OA_5} + \overrightarrow{OA_6})$

It is easy to see that for $t=\frac{1}{2}$ we obtain a point X on this line so that $\overrightarrow{OX} = \frac{1}{2}\cdot\sum_{i=1}^{6}\overrightarrow{OA_i}$.

By analogy, all the partitions A_i, A_j, A_k and A_l, A_m, A_n give straight lines which passes through the same point X.

b). Let H_{123} and G_{456} the orthocenter and the centroid of the triangles A_{123} and A_{456}. The straight line $H_{123}G_{456}$ has the vectorial ecuation: $\overrightarrow{OP} = (1-t)\overrightarrow{OH_{123}} + t\cdot\overrightarrow{OG_{456}}$

Hence $\overrightarrow{OP} = (1-t)(\overrightarrow{OA_1} + \overrightarrow{OA_2} + \overrightarrow{OA_3}) + t\cdot\frac{1}{3}(\overrightarrow{OA_4} + \overrightarrow{OA_5} + \overrightarrow{OA_6})$

It is easy to see that for $t=\frac{3}{4}$ we obtain the point Y on this line so that $\overrightarrow{OY} = \frac{1}{4}\cdot\sum_{i=1}^{6}\overrightarrow{OA_i}$.

By analogy, all the partitions A_i, A_j, A_k and A_l, A_m, A_n give straight lines which passes through th same point Y.

c). $\overrightarrow{OX} = 2\cdot\overrightarrow{OY} \Rightarrow O, X, Y$ collinear and Y is the midpoint of $[OX]$.