Problem 715

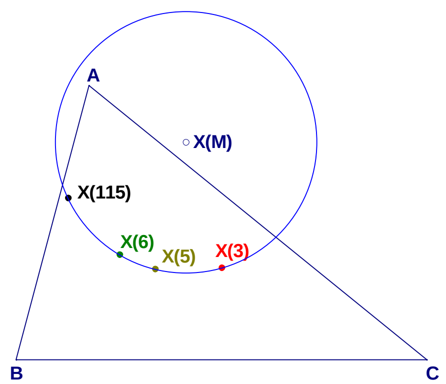
The circumcenter, the nine-point center, the symmedian point (Lemoine) and the center of Kiepert hyperbola lie in a circumference.

Montes, J.(2014): Personal communication.*

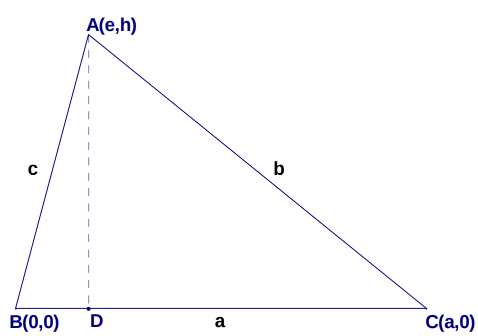
José Montes Valderrama
Teacher at «Triana» Public School for Adults (Sevilla)

1. Introduction

It is going to be proven by analytic geometry that in every triangle ABC, points X(3) (circumcenter), X(5) (center of the nine-point circle), X(6) (Lemoine point) and X(115) (center of Kiepert hyperbola) are conciclic. It will be considered that the radio of the circumference will be zero when the triangle ABC is equilateral and infinite if it is isosceles.



2. Reference triangle



*I would like to extend my sincerest thanks and appreciation to Ricardo Barroso Campos for the publication of this problem

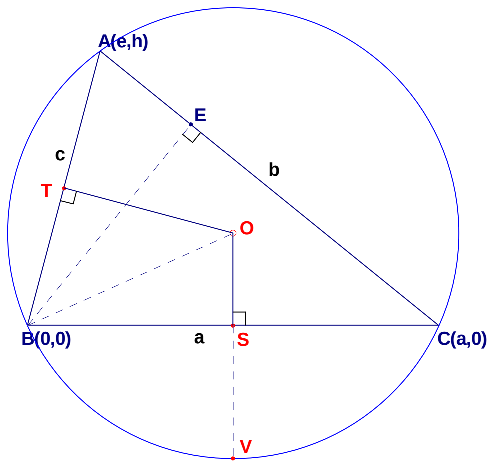
We locate in B the coordinates origin. Applying the law of cosines

$$b^2 = a^2 + c^2 - 2ac\cos B = a^2 + c^2 - 2ac\frac{BD}{c} \Rightarrow BD = e = \frac{a^2 - b^2 + c^2}{2a}$$

and in the rectangular triangle ABD,

$$(AD)^2 = (AB)^2 - (BD)^2 \Rightarrow AD = h = \frac{\sqrt{-a^4 + 2a^2b^2 - b^4 + 2a^2c^2 + 2b^2c^2 - c^4}}{2a}$$

3. Circumcenter coordinates



S is the middle point of BC and T is the middle point of AB. Through these two points, we draw the two perpendicular bisectors of the sides that coincide in the point O. We also draw the height $BE = \frac{ah}{b}$ Using the law of cosines in the triangle ABC

$$a^2 = b^2 + c^2 - 2bc\cos A = b^2 + c^2 - 2bc\frac{AE}{c} \Rightarrow AE = \frac{-a^2 + b^2 + c^2}{2b}$$

it is verified that the measure of the inscribed angle $\widehat{CAB}$ is the half of the arc $\widehat{BC}$ .The arc $\widehat{BC}$ is the same as the arc $\widehat{BV}$ But the central angle $\widehat{SOB}$ is also the arc $\widehat{BV}$, reason that shows that triangles EAB and SOB are similar. By this

$$\frac{OS}{BS} = \frac{AE}{BE}$$

in consequence

$$OS = \frac{(\frac{1}{2}a)(\frac{-a^2+b^2+c^2}{2b})}{\frac{ah}{b}} = \frac{-a^2 + b^2 + c^2}{4h} \qquad \text{and the coordinates are} \qquad \left(\frac{a}{2}\ ,\ \frac{-a^2 + b^2 + c^2}{4h}\right)$$

multiplying, in the ordinate, numerator y denominator by h and having in account the value of h^2:

CIRCUMCENTER COORDINATE x

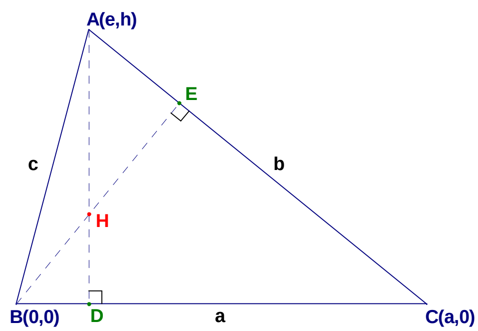
$$O_x = \frac{a}{2}$$

CIRCUMCENTER COORDINATE y

$$O_y = \frac{a^2h(a^2-b^2-c^2)}{a^4-2a^2b^2+b^4-2a^2c^2-2b^2c^2+c^4}$$

4. Coordinates of the nine-point circle

This point will be calculated having in account that it lies in the Euler line and is equidistant to the circumcenter and orthocenter, therefore, we need the orthocenter coordinates



Similarly to previous calculus, $CE = \dfrac{a^2+b^2-c^2}{2b}$, and the similar triangles BCE y DHB implies that

$$\frac{CE}{BE} = \frac{HD}{BD} \Rightarrow HD = \frac{(\frac{a^2+b^2-c^2}{2b})(\frac{a^2-b^2+c^2}{2a})}{\frac{ah}{b}} = \frac{a^4-b^4+2b^2c^2-c^4}{4a^2h}$$

and the orthocenter coordinates are $\left(\dfrac{a^2-b^2+c^2}{2a}\ ,\ \dfrac{a^4-b^4+2b^2c^2-c^4}{4a^2h}\right)$

The middle point between the circumcenter and the orthocenter $\left(\dfrac{2a^2-b^2+c^2}{4a}\ ,\ \dfrac{a^2b^2-b^4+a^2c^2+2b^2c^2-c^4}{8a^2h}\right)$ is the center of the nine-point circle.

Multiplying, in the ordinate, numerator and denominator by h and having in account the value of h^2:

COORDINATE x OF THE CENTER OF THE NINE-POINT CIRCLE

$$N_x = \frac{2a^2-b^2+c^2}{4a}$$

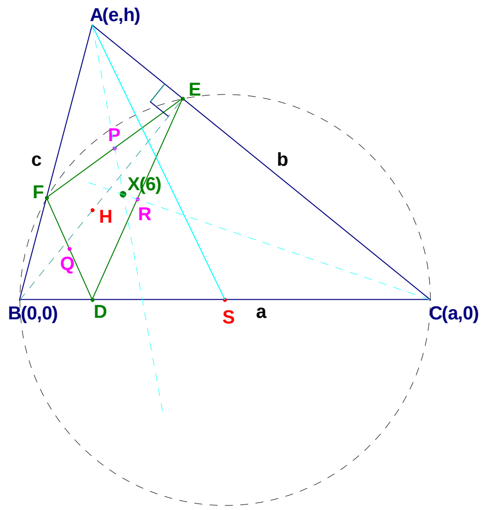
COORDINATE y OF THE CENTER OF THE NINE-POINT CIRCLE

$$N_y = \frac{h(a^2b^2-b^4+a^2c^2+2b^2c^2-c^4)}{2(-a^4+2a^2b^2-b^4+2a^2c^2+2b^2c^2-c^4)}$$

5. Coordinates of Lemoine point

As it is known, Lemoine point is the point of concurrence of the symmedians (reflections of medians in corresponding angle bisectors). To calculate the coordinates of Lemoine point, suppose:

The triangle ABC and its orthic triangle DEF. Each symmedian intersects its side of the orthic triangle through the middle point.



If we draw a circumference with the radio BS and with the center in the middle point S on the BC side, circumference passes through the feet of the altitudes of B and C. This proves that:

$$\widehat{FEA} = \frac{\Pi}{2} - \widehat{BEF} = \frac{\widehat{CB}}{2} - \frac{\widehat{FB}}{2} = \frac{\widehat{CF}}{2} = \widehat{ABC}$$

And the similar triangles ABC y FEA and theirs positions prove that, being P the middle point of FE, the angles $\widehat{PAF}$ y $\widehat{CAS}$ are congruent ,so, the AP line is the symmedian of AS

5.1. Feet of the altitudes *D*, *E*, *F*

Equation of the side AB $\quad hx - ey = 0$

Equation of the side BC $\quad y = 0$

Equation of the side CA $\quad hx - (e-a)y = ha$

Equation of the altitude AD $\quad x = e$

Equation of the altitude BE $\quad (a-e)x - hy = 0$

Equation of the altitude CF $\quad ex + hy = ea$

Point D $\quad D(e\ ,\ 0)$

Point E $\begin{cases} hx - (e-a)y = ha \\ (a-e)x - hy = 0 \end{cases}$ $\quad E\left(\dfrac{h^2a}{b^2}\ ,\ \dfrac{ha(a-e)}{b^2}\right)$

Point F $\begin{cases} hx - ey = 0 \\ ex + hy = ea \end{cases}$ $\quad F\left(\dfrac{ae^2}{c^2}\ ,\ \dfrac{aeh}{c^2}\right)$

5.2. Middle points Q, R of the sides of the orthic triangle

$$Q\left(\frac{ae^2}{2c^2}+\frac{e}{2}\,,\,\frac{aeh}{2c^2}\right) \qquad\qquad R\left(\frac{a^2h}{2b^2}+\frac{e}{2}\,,\,\frac{ah(a-e)}{2b^2}\right)$$

5.3. Intersection of the symmedians BQ, CR

The system made by these two symmedians

$$\begin{cases} BQ \qquad y=\dfrac{2ah}{a^2-b^2+3c^2}\;x \\ CR \qquad y=\dfrac{2ah}{-a^2-3b^2+c^2}\;(x-a) \end{cases}$$

have as a result the Lemione point:

COORDINATE x OF LEMOINE POINT

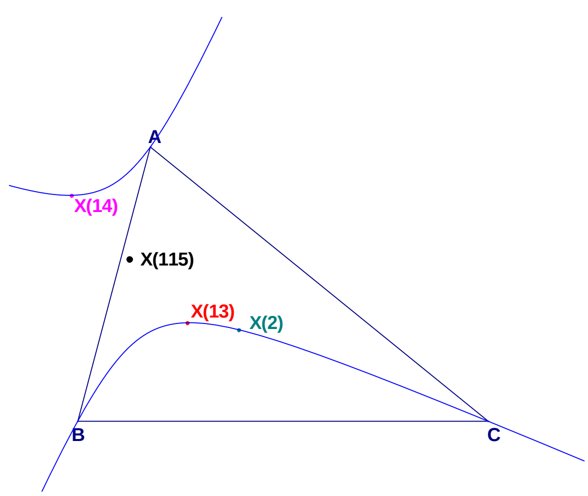
$$L_x=\frac{a(a^2-b^2+3c^2)}{2(a^2+b^2+c^2)}$$

COORDINATE y OF LEMOINE POINT

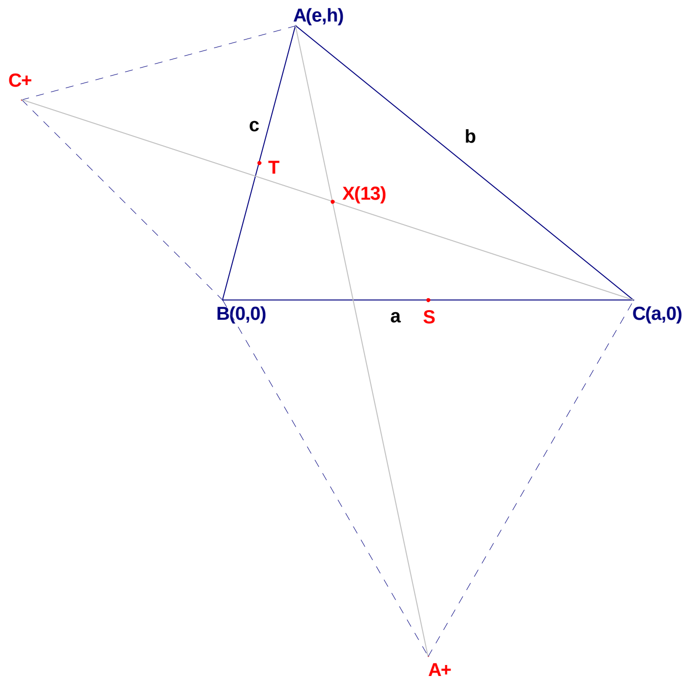
$$L_y=\frac{a^2h}{a^2+b^2+c^2}$$

6. Coordinates of the center of the Kiepert hyperbola

The Kiepert hyperbola of a triangle ABC is the circumscribed rectangular hyperbola passing through the centroid X(2). Also, it passes trough its orthocenter, the Skieper point, Napoleon, Fermat, Vecten and some more points. The center of this hyperbola X(115) is in the middle point of the Fermart points X(13) and X(14) *(C. Kimberling. Encyclopedia of Triangle Centers)*. This property will be used to calculate the center of the Kiepert hyperbola.



6.1. Coordinates of the first point of Fermat



6.1.1. Coordinates A^+

$$A^+\left(\frac{a}{2}\ ,\ \frac{-a\sqrt{3}}{2}\right)$$

6.1.2. Coordinates C^+

Form the system

$$\begin{cases} \dfrac{hx-ey}{-c}=\dfrac{c\sqrt{3}}{2} & \text{distance from the point } C^+ \text{to the line } AB \\ 2ex+2hy=c^2 & \text{line that passes trough } \ T\left(\dfrac{e}{2}\ ,\ \dfrac{h}{2}\right) \text{ and } \perp AB \end{cases}$$

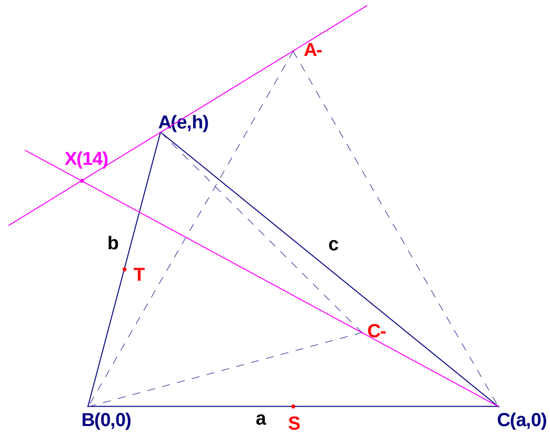
$$C^+\left(\frac{e}{2}-\frac{h\sqrt{3}}{2}\ ,\ \frac{h}{2}+\frac{e\sqrt{3}}{2}\right)$$

6.1.3. Intersection of the lines AA^+, CC^+

$$\begin{cases} AA^+ & (a\sqrt{3}+2h)x+(a-2e)y=a(h+e\sqrt{3}) \\ CC^+ & (h+e\sqrt{3})x+(2a+h\sqrt{3}-e)y=a(h+e\sqrt{3}) \end{cases}$$

$$X(13)\left(\frac{a(h+e\sqrt{3})(a+e+h\sqrt{3})}{6ah+(a^2+b^2+c^2)\sqrt{3}}\ ,\ \frac{a(h+e\sqrt{3})(h+(a-e)\sqrt{3})}{6ah+(a^2+b^2+c^2)\sqrt{3}}\right)$$

6.2. Coordinates of the second point of Fermat



6.2.1. Coordinates A^-

$$A^-\left(\frac{a}{2}\ ,\ \frac{a\sqrt{3}}{2}\right)$$

6.2.2. Coordinates C^-

From the system $\begin{cases} \dfrac{hx-ey}{c}=\dfrac{c\sqrt{3}}{2} & \text{distance from the point } C^- \text{to the line } AB \\ 2ex+2hy=c^2 & \text{line that passes through } T\left(\dfrac{e}{2}\ ,\ \dfrac{h}{2}\right) \text{ and } \perp AB \end{cases}$

$$C^-\left(\frac{e}{2}+\frac{h\sqrt{3}}{2}\ ,\ \frac{h}{2}-\frac{e\sqrt{3}}{2}\right)$$

6.2.3. Intersection of the lines AA^-, CC^-

$$\begin{cases} AA^- & (a\sqrt{3}-2h)x+(2e-a)y=a(e\sqrt{3}-h) \\ CC^- & (e\sqrt{3}-h)x+(e+h\sqrt{3}-2a)y=a(e\sqrt{3}-h) \end{cases}$$

$$X(14)\left(\frac{a(e\sqrt{3}-h)(-a-e+h\sqrt{3})}{6ah-(a^2+b^2+c^2)\sqrt{3}}\ ,\ \frac{a(e\sqrt{3}-h)((a-e)\sqrt{3}-h)}{6ah-(a^2+b^2+c^2)\sqrt{3}}\right)$$

6.3. Middle point between X(13) and X(14)

Abscissa X(115)=$\dfrac{\frac{a(h+e\sqrt{3})(a+e+h\sqrt{3})}{6ah+(a^2+b^2+c^2)\sqrt{3}}+\frac{a(e\sqrt{3}-h)(-a-e+h\sqrt{3})}{6ah-(a^2+b^2+c^2)\sqrt{3}}}{2}=\dfrac{a[3(a^2+b^2+c^2)(-2ae-2c^2)+12h^2(4ae+a^2)]}{24(a^2b^2+b^2c^2+a^2c^2-a^4-b^4-c^4)}$

Ordinate X(115)=$\dfrac{\frac{a(h+e\sqrt{3})(h+(a-e)\sqrt{3})}{6ah+(a^2+b^2+c^2)\sqrt{3}}+\frac{a(e\sqrt{3}-h)((a-e)\sqrt{3}-h)}{6ah-(a^2+b^2+c^2)\sqrt{3}}}{2}=\dfrac{6a^2h[2c^2+6ae-8e^2-(a^2+b^2+c^2)]}{24(a^2b^2+b^2c^2+a^2c^2-a^4-b^4-c^4)}$

having in account the value of e and h^2:

COORDINATE x OF THE CENTER OF KIEPERT HYPERBOLA

$$K_x=\frac{-2a^6+4a^4b^2-3a^2b^4+b^6+2a^2b^2c^2-3b^4c^2-a^2c^4+3b^2c^4-c^6}{4a(-a^4+a^2b^2-b^4+a^2c^2+b^2c^2-c^4)}$$

COORDINATE y OF THE CENTER OF KIEPERT HYPERBOLA

$$K_y=\frac{-h(b^4-2b^2c^2+c^4)}{2(-a^4+a^2b^2-b^4+a^2c^2+b^2c^2-c^4)}$$

7. Center of the circumference X(3)-X(5)-X(6)-X(115)

The center of the circumference will be calculated using the intersection of the two perpendicular bisector of the sides [X(3)X(5)] and [X(3)X(6)]

7.1. Perpendicular bisector of the side [X(3) X(5)]

Middle point between X(3) and X(5) $\left(\dfrac{4a^2-b^2+c^2}{8a}\ ,\ \dfrac{-a^2+b^2+c^2+4h^2}{16h}\right)$

Slope of the perpendicular bisector of the side [X(3) X(5)]=$\dfrac{2h(b^2-c^2)}{a(3a^2-3b^2-3c^2+4h^2)}$

Equation of the perpendicular bisector of the side [X(3) X(5)]
$32ah^2(b^2-c^2)x-16a^2h(3a^2-3b^2-3c^2+4h^2)y=$
$-a^2(3a^2-3b^2-3c^2+4h^2)(-a^2+b^2+c^2+4h^2)+4h^2(b^2-c^2)(4a^2-b^2+c^2)$

7.2. Perpendicular bisector of the side [X(3) X(6)]

Middle point between X(3) and X(6) $\left(\dfrac{a(a^2+2c^2)}{2(a^2+b^2+c^2)}\ ,\ \dfrac{(-a^2+b^2+c^2)(a^2+b^2+c^2)+4a^2h^2}{8h(a^2+b^2+c^2)}\right)$

Slope of the perpendicular bisector of the side [X(3) X(6)]=$\dfrac{4ah(b^2-c^2)}{4a^2h^2-(-a^2+b^2+c^2)(a^2+b^2+c^2)}$

Equation of the perpendicular bisector of the side [X(3) X(6)]
$32ah^2(a^2+b^2+c^2)(b^2-c^2)x-8h[4a^2h^2-(-a^2+b^2+c^2)(a^2+b^2+c^2)](a^2+b^2+c^2)y=$
$[4a^2h^2-(-a^2+b^2+c^2)(a^2+b^2+c^2)][4a^2h^2+(-a^2+b^2+c^2)(a^2+b^2+c^2)]-16a^2h^2(a^2+2c^2)(b^2-c^2)$

Solving the system made by these two perpendicular bisectors of the sides and replacing the value of h^2 we obtain the coordinates of the center of the circumference:

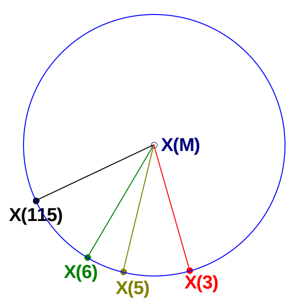
COORDINATE x OF THE CENTER X(M) OF THE CIRCUMFERENCE

$$M_x = \frac{a(-a^{10}b^2+3a^8b^4-2a^6b^6-2a^4b^8+3a^2b^{10}-b^{12}+3a^{10}c^2-8a^8b^2c^2+17a^6b^4c^2-14a^4b^6c^2+2b^{10}c^2-5a^8c^4}{4(-a^{10}b^2+2a^8b^4-2a^4b^8+a^2b^{10}+a^{10}c^2+3a^6b^4c^2-3a^4b^6c^2-b^{10}c^2}$$
$$\frac{+5a^6b^2c^4-2a^4b^4c^4+7a^2b^6c^4-7b^8c^4-2a^6c^6-2a^4b^2c^6-5a^2b^4c^6+16b^6c^6+6a^4c^8-15b^4c^8-a^2c^{10}+6b^2c^{10}-c^{12})}{-2a^8c^4-3a^6b^2c^4+3a^2b^6c^4+2b^8c^4+3a^4b^2c^6-3a^2b^4c^6+2a^4c^8-2b^4c^8-a^2c^{10}+b^2c^{10})}$$

COORDINATE y OF THE CENTER X(M) OF THE CIRCUMFERENCE

$$M_y = \frac{h(-a^{10}+2a^8b^2-2a^4b^6+a^2b^8+2a^8c^2-a^6b^2c^2-a^4b^4c^2-2a^2b^6c^2-a^4b^2c^4}{-2a^{10}+4a^8b^2-4a^4b^6+2a^2b^8+4a^8c^2+6a^6b^2c^2-10a^4b^4c^2+2a^2b^6c^2}$$
$$\frac{+6a^2b^4c^4-2a^4c^6-2a^2b^2c^6+a^2c^8)}{-2b^8c^2-10a^4b^2c^4+8a^2b^4c^4+2b^6c^4-4a^4c^6+2a^2b^2c^6+2b^4c^6+2a^2c^8-2b^2c^8}$$

8. Equality of the radios



Calculating the squared radios

$$\begin{aligned}
\text{RADIO}^2[\text{X(M)X(3)}] &= (M_x-O_x)^2+(M_y-O_y)^2\\
\text{RADIO}^2[\text{X(M)X(5)}] &= (M_x-N_x)^2+(M_y-N_y)^2\\
\text{RADIO}^2[\text{X(M)X(6)}] &= (M_x-L_x)^2+(M_y-L_y)^2\\
\text{RADIO}^2[\text{X(M)X(115)}] &= (M_x-K_x)^2+(M_y-K_y)^2
\end{aligned}$$

it is necessary to show the equality of the last (because the fourth point X(115) has not been used in the calculation of the center) with any of the first three to verify that the circumference passes through the four points.Therefore

$$\text{RADIO}^2[\text{X(M)X(3)}] = \text{RADIO}^2[\text{X(M)X(115)}]$$

then

$$[((M_x-O_x)^2+(M_y-O_y)^2]-[(M_x-K_x)^2+(M_y-K_y)^2]=0$$

$$[((M_x-O_x)^2+(M_y-O_y)^2]-[(M_x-K_x)^2+(M_y-K_y)^2]=$$

$$(4a^{24}b^2-22a^{22}b^4+50a^{20}b^6-55a^{18}b^8+14a^{16}b^{10}+42a^{14}b^{12}-62a^{12}b^{14}+40a^{10}b^{16}-10a^8b^{18}-4a^6b^{20}+
4a^4b^{22}-a^2b^{24}+4a^{24}c^2-52a^{22}b^2c^2+186a^{20}b^4c^2-344a^{18}b^6c^2+418a^{16}b^8c^2-393a^{14}b^{10}c^2+305a^{12}b^{12}c^2-
167a^{10}b^{14}c^2+27a^8b^{16}c^2+45a^6b^{18}c^2-43a^4b^{20}c^2+15a^2b^{22}c^2-b^{24}c^2-22a^{22}c^4+186a^{20}b^2c^4-466a^{18}b^4c^4+
588a^{16}b^6c^4+444a^{14}b^8c^4+211a^{12}b^{10}c^4-102a^{10}b^{12}c^4+159a^8b^{14}c^4-226a^6b^{16}c^4+191a^4b^{18}c^4-84a^2b^{20}c^4+
9b^{22}c^4+50a^{20}c^6-344a^{18}b^2c^6+588a^{16}b^4c^6-458a^{14}b^6c^6+174a^{12}b^8c^6+55a^{10}b^{10}c^6-267a^8b^{12}c^6+440a^6b^{14}c^6-
422a^4b^{16}c^6+251a^2b^{18}c^6-35b^{20}c^6-55a^{18}c^8+418a^{16}b^2c^8-444a^{14}b^4c^8+174a^{12}b^6c^8-84a^{10}b^8c^8+123a^8b^{10}c^8-
458a^6b^{12}c^8+490a^4b^{14}c^8-471a^2b^{16}c^8+75b^{18}c^8+14a^{16}c^{10}-393a^{14}b^2c^{10}+211a^{12}b^4c^{10}+55a^{10}b^6c^{10}+
123a^8b^8c^{10}+406a^6b^{10}c^{10}-220a^4b^{12}c^{10}+630a^2b^{14}c^{10}-90b^{16}c^{10}+42a^{14}c^{12}+305a^{12}b^2c^{12}-102a^{10}b^4c^{12}-
267a^8b^6c^{12}-458a^6b^8c^{12}-220a^4b^{10}c^{12}-680a^2b^{12}c^{12}+42b^{14}c^{12}-62a^{12}c^{14}-167a^{10}b^2c^{14}+159a^8b^4c^{14}+
440a^6b^6c^{14}+490a^4b^8c^{14}+630a^2b^{10}c^{14}+42b^{12}c^{14}+40a^{10}c^{16}+27a^8b^2c^{16}-226a^6b^4c^{16}-422a^4b^6c^{16}-
471a^2b^8c^{16}-90b^{10}c^{16}-10a^8c^{18}+45a^6b^2c^{18}+191a^4b^4c^{18}+251a^2b^6c^{18}+75b^8c^{18}-4a^6c^{20}-43a^4b^2c^{20}-
84a^2b^4c^{20}-35b^6c^{20}+4a^4c^{22}+15a^2b^2c^{22}+9b^4c^{22}-a^2c^{24}-b^2c^{24}+16a^{22}b^2h^2-56a^{20}b^4h^2+72a^{18}b^6h^2-
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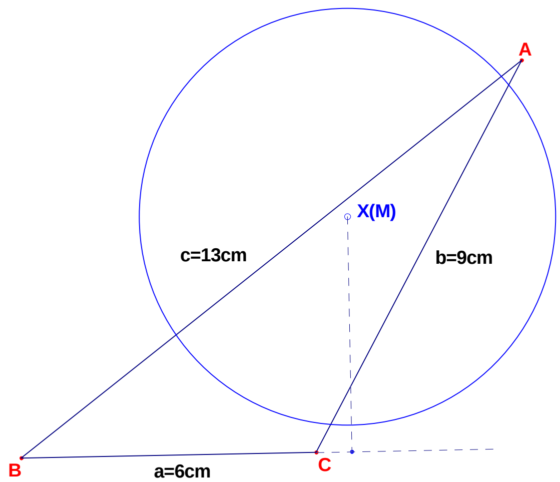
72a^6b^{16}c^2h^2+44a^4b^{18}c^2h^2-4a^2b^{20}c^2h^2-56a^{20}c^4h^2+360a^{18}b^2c^4h^2-600a^{16}b^4c^4h^2+480a^{14}b^6c^4h^2-
244a^{12}b^8c^4h^2+180a^{10}b^{10}c^4h^2-264a^8b^{12}c^4h^2+280a^6b^{14}c^4h^2-196a^4b^{16}c^4h^2+28a^2b^{18}c^4h^2+72a^{18}c^6h^2-
480a^{16}b^2c^6h^2+480a^{14}b^4c^6h^2-152a^{12}b^6c^6h^2-72a^{10}b^8c^6h^2+432a^8b^{10}c^6h^2-440a^6b^{12}c^6h^2+464a^4b^{14}c^6h^2-
80a^2b^{16}c^6h^2-20a^{16}c^8h^2+456a^{14}b^2c^8h^2-244a^{12}b^4c^8h^2-72a^{10}b^6c^8h^2-440a^8b^8c^8h^2+224a^6b^{10}c^8h^2-
696a^4b^{12}c^8h^2+112a^2b^{14}c^8h^2-56a^{14}c^{10}h^2-332a^{12}b^2c^{10}h^2+180a^{10}b^4c^{10}h^2+432a^8b^6c^{10}h^2+224a^6b^8c^{10}h^2+
776a^4b^{10}c^{10}h^2-56a^2b^{12}c^{10}h^2+76a^{12}c^{12}h^2+140a^{10}b^2c^{12}h^2-264a^8b^4c^{12}h^2-440a^6b^6c^{12}h^2-696a^4b^8c^{12}h^2-
56a^2b^{10}c^{12}h^2-40a^{10}c^{14}h^2+16a^8b^2c^{14}h^2+280a^6b^4c^{14}h^2+464a^4b^6c^{14}h^2+112a^2b^8c^{14}h^2+4a^8c^{16}h^2-
72a^6b^2c^{16}h^2-196a^4b^4c^{16}h^2-80a^2b^6c^{16}h^2+8a^6c^{18}h^2+44a^4b^2c^{18}h^2+28a^2b^4c^{18}h^2-4a^4c^{20}h^2-4a^2b^2c^{20}h^2)/
[16a^2(a^2-c^2)(a^4-b^4+a^2c^2-b^2c^2)(a^4-2a^2b^2+b^4-2a^2c^2-2b^2c^2+c^4)^2(a^4-a^2b^2+b^4-a^2c^2-b^2c^2+c^4)^2]$$

Taking as common factor $4a^2h^2$ in the summed numbers of the numerator, will be replaced by its value $-a^4+2a^2b^2-b^4+2a^2c^2+2b^2c^2-c^4$, and calculating the products

$$4a^{24}b^2-22a^{22}b^4+50a^{20}b^6-55a^{18}b^8+14a^{16}b^{10}+42a^{14}b^{12}-62a^{12}b^{14}+40a^{10}b^{16}-10a^8b^{18}-4a^6b^{20}+
4a^4b^{22}-a^2b^{24}+4a^{24}c^2-52a^{22}b^2c^2+186a^{20}b^4c^2-344a^{18}b^6c^2+418a^{16}b^8c^2-393a^{14}b^{10}c^2+305a^{12}b^{12}c^2-
167a^{10}b^{14}c^2+27a^8b^{16}c^2+45a^6b^{18}c^2-43a^4b^{20}c^2+15a^2b^{22}c^2-b^{24}c^2-22a^{22}c^4+186a^{20}b^2c^4-466a^{18}b^4c^4+
588a^{16}b^6c^4+444a^{14}b^8c^4+211a^{12}b^{10}c^4-102a^{10}b^{12}c^4+159a^8b^{14}c^4-226a^6b^{16}c^4+191a^4b^{18}c^4-84a^2b^{20}c^4+
9b^{22}c^4+50a^{20}c^6-344a^{18}b^2c^6+588a^{16}b^4c^6-458a^{14}b^6c^6+174a^{12}b^8c^6+55a^{10}b^{10}c^6-267a^8b^{12}c^6+440a^6b^{14}c^6-
422a^4b^{16}c^6+251a^2b^{18}c^6-35b^{20}c^6-55a^{18}c^8+418a^{16}b^2c^8-444a^{14}b^4c^8+174a^{12}b^6c^8-84a^{10}b^8c^8+123a^8b^{10}c^8-
458a^6b^{12}c^8+490a^4b^{14}c^8-471a^2b^{16}c^8+75b^{18}c^8+14a^{16}c^{10}-393a^{14}b^2c^{10}+211a^{12}b^4c^{10}+55a^{10}b^6c^{10}+
123a^8b^8c^{10}+406a^6b^{10}c^{10}-220a^4b^{12}c^{10}+630a^2b^{14}c^{10}-90b^{16}c^{10}+42a^{14}c^{12}+305a^{12}b^2c^{12}-102a^{10}b^4c^{12}-
267a^8b^6c^{12}-458a^6b^8c^{12}-220a^4b^{10}c^{12}-680a^2b^{12}c^{12}+42b^{14}c^{12}-62a^{12}c^{14}-167a^{10}b^2c^{14}+159a^8b^4c^{14}+
440a^6b^6c^{14}+490a^4b^8c^{14}+630a^2b^{10}c^{14}+42b^{12}c^{14}+40a^{10}c^{16}+27a^8b^2c^{16}-226a^6b^4c^{16}-422a^4b^6c^{16}-
471a^2b^8c^{16}-90b^{10}c^{16}-10a^8c^{18}+45a^6b^2c^{18}+191a^4b^4c^{18}+251a^2b^6c^{18}+75b^8c^{18}-4a^6c^{20}-43a^4b^2c^{20}-
84a^2b^4c^{20}-35b^6c^{20}+4a^4c^{22}+15a^2b^2c^{22}+9b^4c^{22}-a^2c^{24}-b^2c^{24}-4a^{24}b^2+22a^{22}b^4-50a^{20}b^6+55a^{18}b^8-
14a^{16}b^{10}-42a^{14}b^{12}+62a^{12}b^{14}-40a^{10}b^{16}+10a^8b^{18}+4a^6b^{20}-4a^4b^{22}+a^2b^{24}-4a^{24}c^2+52a^{22}b^2c^2-
186a^{20}b^4c^2+344a^{18}b^6c^2-418a^{16}b^8c^2+393a^{14}b^{10}c^2-305a^{12}b^{12}c^2+167a^{10}b^{14}c^2-27a^8b^{16}c^2-45a^6b^{18}c^2+
43a^4b^{20}c^2-15a^2b^{22}c^2+b^{24}c^2+22a^{22}c^4-186a^{20}b^2c^4+466a^{18}b^4c^4-588a^{16}b^6c^4+444a^{14}b^8c^4-211a^{12}b^{10}c^4+
102a^{10}b^{12}c^4-159a^8b^{14}c^4+226a^6b^{16}c^4-191a^4b^{18}c^4+84a^2b^{20}c^4-9b^{22}c^4-50a^{20}c^6+344a^{18}b^2c^6-588a^{16}b^4c^6+
458a^{14}b^6c^6-174a^{12}b^8c^6-55a^{10}b^{10}c^6+267a^8b^{12}c^6-440a^6b^{14}c^6+422a^4b^{16}c^6-251a^2b^{18}c^6+35b^{20}c^6+
55a^{18}c^8-418a^{16}b^2c^8+444a^{14}b^4c^8-174a^{12}b^6c^8+84a^{10}b^8c^8-123a^8b^{10}c^8+458a^6b^{12}c^8-490a^4b^{14}c^8+
471a^2b^{16}c^8-75b^{18}c^8-14a^{16}c^{10}+393a^{14}b^2c^{10}-211a^{12}b^4c^{10}-55a^{10}b^6c^{10}-123a^8b^8c^{10}-406a^6b^{10}c^{10}+
220a^4b^{12}c^{10}-630a^2b^{14}c^{10}+90b^{16}c^{10}-42a^{14}c^{12}-305a^{12}b^2c^{12}+102a^{10}b^4c^{12}+267a^8b^6c^{12}+458a^6b^8c^{12}+
220a^4b^{10}c^{12}+680a^2b^{12}c^{12}-42b^{14}c^{12}+62a^{12}c^{14}+167a^{10}b^2c^{14}-159a^8b^4c^{14}-440a^6b^6c^{14}-490a^4b^8c^{14}-
630a^2b^{10}c^{14}-42b^{12}c^{14}-40a^{10}c^{16}-27a^8b^2c^{16}+226a^6b^4c^{16}+422a^4b^6c^{16}+471a^2b^8c^{16}+90b^{10}c^{16}+10a^8c^{18}-
45a^6b^2c^{18}-191a^4b^4c^{18}-251a^2b^6c^{18}-75b^8c^{18}+4a^6c^{20}+43a^4b^2c^{20}+84a^2b^4c^{20}+35b^6c^{20}-4a^4c^{22}-
15a^2b^2c^{22}-9b^4c^{22}+a^2c^{24}+b^2c^{24}=\mathbf{0}$$

9. Coordinates of the center of Kimberling´s reference triangle

Clark Kimberling has established in his *Encyclopedia of Triangle Centers (ETC)* a system of classification that consists in associating to each characteristic point, the distance from this point to the base BC or its extension, using the following reference triangle



So, the coordinates of the center of the circumference are $\left(\dfrac{29092121893}{4325798400}, \dfrac{9931771\sqrt{35}}{12289200}\right)$, and the ordinate sets the distance

$$\boxed{\frac{9931771\sqrt{35}}{12289200} = 4.7812021631574924911 9}$$