Problema 733.-

E 3155. Demostrar que para cualquier triángulo ABC existen puntos A' B' y C' que satisfacen:

1) A' está en el lado BC, B' en AC y C' en AB.

2) A'C + CB' = B'A + AC' = C'B + BA';

3) AA', BB', y CC' concurren en un punto.

Trisecting a Triangle by Cevians

E 3155 [1986, 482]. *Proposed by Gene Bennett, John Glenn, and Clark Kimberling, University of Evansville.*

Prove that for any triangle ABC, there exist points A', B', C' satisfying
(1) A' lies on side $\overline{BC}$, B' on side $\overline{AC}$, and C' on side $\overline{AB}$.
(2) $\overline{A'C} + \overline{CB'} = \overline{B'A} + \overline{AC'} = \overline{C'B} + \overline{BA'}$; and
(3) $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$ concur in a point.

Composite Solution based on the solutions of Jeffrey M. Cohen and others. Let $\overline{AB}$ be the shortest side. For any choice of C' on $\overline{AB}$, determine points A' and B' on the perimeter of triangle ABC which are one-third of the way around the perimeter from C' in the opposite direction from A and B respectively. (One of these will be on the longest side; the other one will be either on the third side or on the longest side.)

The location of the point I of intersection of $\overline{AA'}$ and $\overline{BB'}$ is a continuous function of C'. It lies on the B side of $\overline{CC'}$ when $C' = A$ and on the A side of $\overline{CC'}$ when $C' = B$. The Intermediate Value Theorem implies that, as C' sweeps through all positions from A to B, there must be at least one position where $\overline{CC'}$ contains I. When this happens, A' will be on the side $\overline{BC}$ and B' will be on the side $\overline{CA}$.

Editorial comment. Most solvers used the 17th-century theorem of Ceva from projective geometry. Suppose a line is drawn from each vertex of a triangle to a point on the opposite side, thus cutting each side into two segments. Ceva's theorem asserts that the product of the lengths of the three segments clockwise from the vertices equals the product of the lengths of the three segments counterclockwise from the vertices if and only if the three lines drawn are concurrent. When the lines are concurrent and the two triple products are equal, the three points on the sides are called "Cevian points." Interestingly, the solvers using Ceva's theorem were almost precisely those *not* from North America; Ceva's theorem seems much better known among mathematical communities outside North America. In the present instance the use of Ceva's Theorem was not an indispensable part of the argument.

Solved by J. M. Cohen, J. Dou (Spain), J. Fukuta (Japan), L. Kuipers (Switzerland), N. Lord (England), O. P. Lossers (The Netherlands), E. Morgantini (Italy), V. Pambuccian (Romania), A. Pedersen (Denmark), W. Raffke (West Germany), V. Schindler (East Germany), L. Smith (student, Canada), J. H. Steelman, P. Tzermias (Greece), P. J. Zwier, and the proposers. Two incorrect solutions were received.