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The Triangle Construction $\{\alpha, b-c, t_A\}$

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Abstract. We study the problem of constructing a triangle from the data $\{\alpha, b-c, t_A\}$, t_A being the length of the internal angle bisector of angle A of a triangle with side lengths a, b, c, and angles α, β, γ. The key-point is the detection of a parabola intimately related to the construction problem.

1. The problem

Denoting, as usual, by $\{a = |BC|, b = |CA|, c = |AB|\}$ the side-lengths, by $\{\alpha, \beta, \gamma\}$ the angles and by $\{t_A, t_C, t_B\}$ the lengths of the internal bisectors of the triangle ABC, the problem at hand is to construct the triangle, given the data $\{\alpha, b-c, t_A\}$. Figure 1 emphasizes the known parts of the triangle assuming that

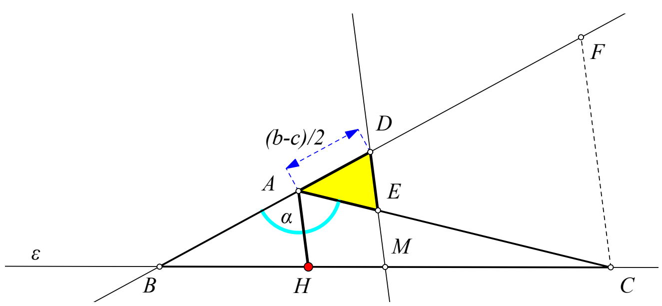

Figure 1. Representing the difference $\frac{b-c}{2}$

$b > c$. The isosceles triangle ADE is created by intersecting the sides $\{AB, AC\}$ with the parallel ME to the bisector AH, from the middle M of the side BC. It is easy to see that, ADE is an isosceles triangle and its lateral sides have the given length $(b-c)/2$. Later, for example, follows by drawing a parallel CF to the bisector AH and noticing that

$$|AD| = |BD| - |AB| = \frac{b+c}{2} - c = \frac{b-c}{2}.$$

In this figure, the known elements are the triangle ADE and the position, relative to ADE, and length of the bisector AH. Thus, the problem reduces to the construction of the appropriate line ε through H, which will define, through its intersections with the lateral sides of ADE, the other two vertices $\{B, C\}$.

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2. The parabola

The three sides of the triangle ABC and the line ME define a parabola ([1, II, p.212]), to which the four lines are tangent, and a key point is, that this parabola is constructible from the given data. Denoting by J the middle of DE and by I the intersection point of the external bisector ζ of the angle $\widehat{A}$ with the medial line of BC, we formulate this property in the following lemma.

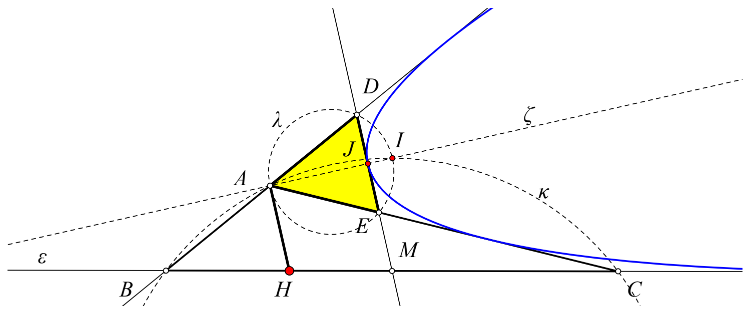

Figure 2. Parabola tangent to $\{AB, BC, CA, DE\}$

Lemma 1. *The parabola, with focus at I and having for tangent at its vertex the line DE, is tangent also to the three sides of the triangle ABC.*

Proof. The lemma results easily from the well known property of parabolas tangent to four lines in general position. It is known that the focus of such a parabola is the intersection point of the circumcircles of the four triangles, formed by three, of the four given lines ([4, p.222]). It suffices here to identify this intersection point with I (See Figure 2), which is trivial. Notice that I is constructible from the given data, since it coincides with the other than A intersection point of the medial line of DE with the circumcircle λ of the triangle ADE. $\square$

3. The solution

Since the parabola is completely defined by the given data, it suffices to construct it and draw from H the tangents to it. This is a ruler and compasses construction, since it only involves the location of intersection points of a parabola and a given line, given the focus and the directrix of the parabola ([5, p. 42]). For the completeness of the exposition, I describe here the construction in a few steps:

(1) Find first the directrix, by drawing the parallel η to DE at the double of its distance from I (See Figure 3).

(2) Find the tangent ξ of the parabola at its intersection point L with the parallel ζ' to its axis ζ from H. This is the intersection point of the line ζ' with the medial line ξ of the segment IN, where N is the intersection point $N = (\zeta', \eta)$.

(3) Draw the parallel ξ' to ξ at double the distance of H from ξ and locate the intersection points $\{P, P'\}$ of the parabola with line ξ'. Lines $\{HP, HP'\}$ are the tangents from H to the parabola, which solve the construction problem.

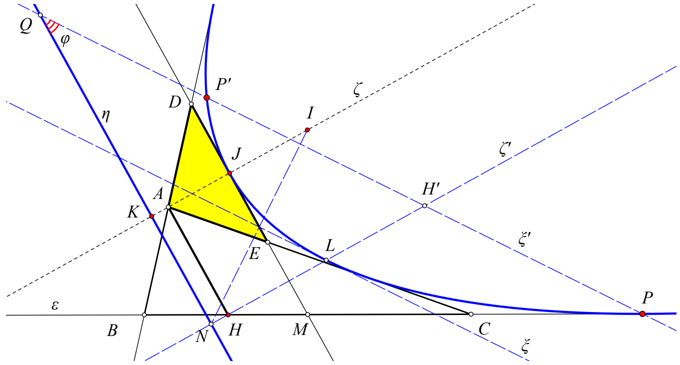

Figure 3. Constructing the tangent to the parabola from H

The intersection points $\{P, P'\}$, of the line ξ' with the parabola, can be constructed, using the angle ϕ of ξ' to η and an Apollonian circle. In fact, consider the intersection point Q of lines $Q = (\xi', \eta)$ and the ratio for arbitrary points X on the line ξ' :

$$\frac{|XI|}{|XQ|} = \frac{|XX_0|}{|XQ|} = \sin(\phi),$$

where X_0 is the projection of X on the directrix η. Hence points $\{P, P'\}$ are the intersections of line ξ' and the Apollonian circle of the segment IQ, for the ratio $k = \sin(\phi)$.

Remarks. (1) The tangent line DE to the parabola was used by Connelly and Randrianantoanina [3] also in some triangle construction problems in another context.

(2) A similar solution can be applied to the construction problem from the elements $\{\alpha, b-c, t'_A\}$, where t'_A denotes the length of the exterior bisector of the angle $\widehat{A}$.

(3) This interpretation of $b-c$ can be used to solve similar construction problems, e.g. from the elements $\{\alpha, b-c, m_A\}$, which is trivial, or $\{\alpha, b-c, h_A\}$, which is more involved ([2, p.144]), m_A and h_A denoting here, respectively, the median and the altitude from A.

References

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