Problema 797

Construcción. Dado el el triángulo ABC, hallar dos puntos D,E sobre el segmento BC tales que AD y AE sean rectas isogonales y el área de ADE
sea la mitad del área de ABC.

García, F. J.(2016): Comunicación personal.

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In order to have $(ADE)=\frac{(ABC)}{2}$ we must have $AD+EC=\frac{a}{2}$ or $\frac{2AD}{a}+\frac{2EC}{a}=1$ (1)

If $D=(0:y:z)$ in barycentric coordinates then $E=\left(0:\frac{b^2}{y}:\frac{c^2}{z}\right)$ and since

$$\frac{AD}{a}=\frac{z}{y+z} \text{ and } \frac{EC}{a}=\frac{\frac{b^2}{y}}{\frac{b^2}{y}+\frac{c^2}{z}}=\frac{b^2z}{b^2z+c^2y}$$

from (1) we get $c^2y^2+(b^2+c^2)yz-3b^2z^2=0$

and if we put $\frac{DC}{BD}=\frac{y}{z}=u$ and $b^2=mc$ then the above equation becomes

$cu^2+(m+c)u-3\frac{b^2}{c}=0$ with a unique solution $u=\frac{\frac{m+c}{2}+\sqrt{\left(\frac{m+c}{2}\right)^2+3b^2}}{c}$.

So the line AD meets the parallel from C to AB at a point *N* such that *CN* = uc since by Thales theorem $\frac{CH}{AB}=\frac{DC}{BD}=u$.

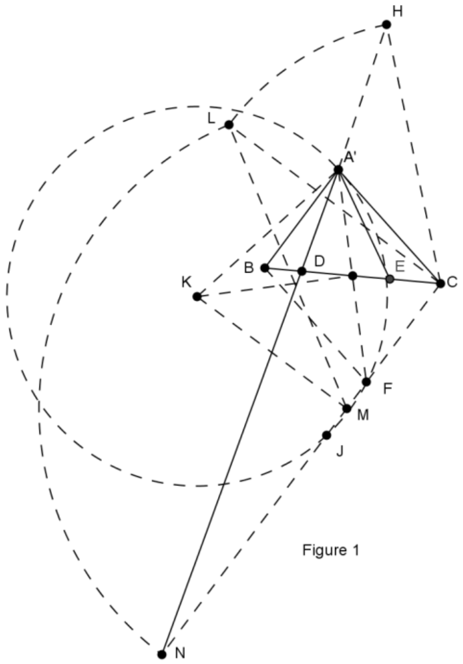

Figure 1

Construction of *CN*. (Figure 1)

Construct the parallelogram *ABFC*. The perpendicular bisector of AF meets the perpendicular to *AC* at *A* at the point *K*. The circle (K, KA) is tangent to *AC* passes through A, F and meets the line *CF* at the point *J*. The orthogonal projection of *K* on the line *CF* is the midpoint *M* of *FJ* where since $CF = c$ and $CA^2 = CF.CJ \Rightarrow b^2 = c.CJ$ we conclude that $CJ = m$ and $CM = \dfrac{m+c}{2}$.

The rotation of *C* around *A* by an angle $\dfrac{2\pi}{3}$ gives the point *H* such that $CH = b\sqrt{3}$.

On the perpendicular to *CM* at *C* we take the segment *CL*=*CH* and on the extension of *CM* we take the point *N* such that $MN = ML = \sqrt{CM^2 + CL^2} = \sqrt{\left(\dfrac{m+c}{2}\right)^2 + 3b^2}$

and hence $CN = \dfrac{m+c}{2} + \sqrt{\left(\dfrac{m+c}{2}\right)^2 + 3b^2}$. The line *AN* meets the line *BC* at the required point *D*. It is easy now to construct the line *AE* isogonal to *AD*.