Similarly,

$$\frac{b}{c+a}(\cos C + \cos A) = 1 - \cos B,$$
$$\frac{c}{a+b}(\cos A + \cos B) = 1 - \cos C.$$

Putting these into (4), we have

$$\frac{a_1}{a} + \frac{b_1}{b} + \frac{c_1}{c} \geq \cos A + \cos B + \cos C = 1 + \frac{r}{R}.$$

Editorial comment. Peter Nüesch (Switzerland) notes that this problem may be viewed as a special case of Problem 1320 in *Mathematics Magazine*, proposedby V. Kovner in vol. 62 (1989), p. 137, solved by J. Heuver and Richard E. Pfiefer in vol. 63(1990) pp. 130–131.

Also solved by P. P. Dályay (Hungary), P. Nüesch (Switzerland), J. Posch, R. Stong, and the proposer.

Triangle Center $X(79)$

11554 [2011, 178]. *Proposed by Zhang Yun, Xi'an Jiao Tong University Sunshine High School, Xi'an, China.* In triangle ABC, let I be the incenter, and let A', B', C' be the reflections of I through sides BC, CA, AB, respectively. Prove that the lines AA', BB', and CC' are concurrent.

Solution by Alin Bostan, INRIA, Rocquencourt, France. First we identify this problem as a particular case of two different classical theorems in Euclidean geometry: Jacobi's Theorem and Kariya's Theorem (which is itself a particular case of an older theorem of Lemoine's, see below). We then give two proofs of Problem 11554.

Jacobi's Theorem (sometimes called "the Isogonal Theorem"): *If ABC is a triangle, and A', B', and C' are points in its plane such that $\angle B'AC = \angle BAC'$, $\angle C'BA = \angle CBA'$, and $\angle A'CB = \angle ACB'$, then the lines AA', BB', and CC' are concurrent.* This is a generalization of the famous "Napoleon's Theorem", available at `http://en.wikipedia.org/wiki/Napoleon's_theorem`. It was seemingly discovered by Carl Friedrich Andreas Jacobi [not to be confused with Carl Gustav Jacob Jacobi], and published in 1825 in Latin: C. F. A. Jacobi, *De triangulorum rectilineorum proprietatibus quibusdam nondum satis cognitis,* Naumburg (1825).

Kariya's Theorem: *Let I be the incenter of a triangle ABC, and let X, Y, Z be the points where the incircle of $\triangle ABC$ touches the sides BC, CA, AB, respectively. If A', B', C' are three points on the half-lines IX, IY, IZ, respectively, such that $IA' = IB' = IC'$, then the lines AA', BB', and CC' are concurrent.* This theorem has a long history. It was discovered independently by Auguste Boutin and by V. Retali: A. Boutain, "Sur un groupe de quatre coniques remarquables,"*Journal de mathématiques spéciales* ser. 3, **4** (1890) 104–107, 124–127; A. Boutin, "Problèmes sur le triangle,"*Journal de mathématiques spéciales* ser. 3, **4** (1890) 265–269; V. Retali, *Periodico di Matematica* (Rome) **11** (1896) 71.

The result only became well known with Kariya's paper (which inspired many results appearing in *l'Enseignement* over the following years): J. Kariya, "Un probléme sur le triangle,"*L'Enseignement mathématique* **6** (1904) 130–132, 236, 406. Actually, a generalization of this result was obtained before Kariya by Emile Lemoine in Section 4 of: E. Lemoine, "Contributions à la géométrie du triangle,"*Congrès de l'AFAS,* Paris, 1889, p. 197–222.

Lemoine explicitly states and proves on page 202 the following: *Let ABC be a triangle, M a point in its plane, and X, Y, Z the projections of M on BC, CA, AB, respectively. If A', B', C' are points on the half-lines MX, MY, and MZ, respectively, such that $MX \cdot MA' = MY \cdot MB' = MZ \cdot MC'$, then AA', BB', CC' are concurrent.*
Auric gave in 1915 another generalization of Kariya's Theorem: A. Auric, "Généralisation du théorème de Kariya,"*Nouvelles annales de mathématiques* 4e série**15** (1915) 222–225. The statement is the same as Lemoine's Theorem except that the assumption $MX \cdot MA' = MY \cdot MB' = MZ \cdot MC'$ is replaced by $MX/MA' = MY/MB' = MZ/MC'$.

Now we give the two solutions to Problem 11554, both based on Ceva's Theorem.

(1) This solution is possibly new (less elegant than the second one, but a bit shorter). Let P be the intersection of AA' and BC, and let Q be the intersection of AI and BC. Applying Menelaus' Theorem twice (once for $\triangle APQ$ and transversal IA', once for $\triangle AIA'$ and transversal BC), we find that $BP/PC = (a^2 + c^2 - b^2 + ca)/(b^2 + a^2 - c^2 + ab)$. Since the numerator is obtained from the denominator by the cyclic permutation $a \to b \to c \to a$, the conclusion follows from Ceva's Theorem.

(2) The second solution is much more elegant, and is possibly due to the Romanian geometer Gheorghe Titeica (it appears as Problem 1138 in his book *Problems of Geometry* (in Romanian)). Let the parallel to BC passing through A' intersect AB and AC in A_1 and A_2, respectively. Construct similarly the points B_1, B_2, C_1, and C_2. By symmetry, $A'A_1 = C'C_2$, $A'A_2 = B'B_1$, and $B'B_2 = C'C_1$. Let P be the intersection of AA' and BC, let Q be the intersection of BB' and AC, and let R be the intersection of CC' and AB. Thales' Theorem implies $BP/PC = A_1A'/A'A_2$, $CQ/QA = B_1B'/B'B_2$, and $AR/RB = C_1C'/C'C_2$. It follows that

$$\frac{BP}{PC}\,\frac{CQ}{QA}\,\frac{AR}{RB} = \frac{A_1A'}{A'A_2}\,\frac{B_1B'}{B'B_2}\,\frac{C_1C'}{C'C_2} = 1,$$

and the conclusion follows from Ceva's Theorem.

Final notes: (i) Nowadays the point J of concurrence in Problem 11554 is sometimes called "Gray's point" after Steve Gray who noted a seemingly new property, namely that the line IJ is parallel to the Euler line OH of $\triangle ABC$.

(ii) The point J is called $X(79)$ in Kimberling's *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

Also solved by Y. An (China), G. Apostolopoulos (Greece), M. Bataille (France), R. B. Campos (Spain), C. Curtis, P. P. Dályay (Hungary), P. De (India), C. Delorme (France), A. Ercan (Turkey), O. Faynshteyn (Germany), R. Frank & H. Riede (Germany), O. Geupel (Germany), J.-P. Grivaux (France), E. A. Herman, S. Hitotumatu (Japan), Y. J. Ionin, M. E. Kidwell & M. D. Meyerson, O. Kouba (Syria), R. Mabry, R. Murgatroyd, C. R. Pranesachar (India), J. Schlosberg, T. Smith, R. Stong, M. Tetiva (Romania), R. S. Tiberio, Z. Vörös (Hungary), Z. Xintao (China), P. Yff, J. B. Zacharias, D. Zeilberger, GCHQ Problem Solving Group (U. K.), and the proposer.

Value Defined by an Integral

11555 [2011, 178]. *Proposed by Duong Viet Thong, National Economics University, Hanoi, Vietnam.* Let f be a continuous real-valued function on $[0, 1]$ such that $\int_0^1 f(x)\,dx = 0$. Prove that there exists c in the interval $(0, 1)$ such that $c^2 f(c) = \int_0^c (x + x^2) f(x)\,dx$.

Solution I by Michael W. Botsko, Saint Vincent College, PA. First, let $F(x) = x\int_0^x f(t)\,dt - \int_0^x t f(t)\,dt$ on $[0, 1]$. By its construction, $F'(x) = \int_0^x f(t)\,dt$ and