NOTE GÉOMÉTRIQUE

PROBLEM 2

50th KÜRSCHÁK MATHEMATICS COMPETITION

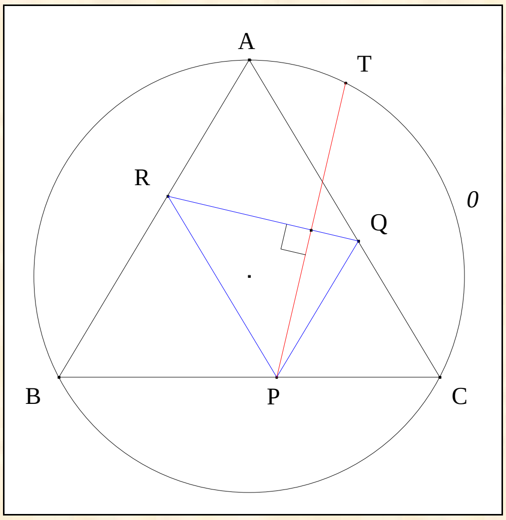
ORGANIZED

BY

JÁNOS BOLYAI MATHEMATICAL SOCIETY

1949

Jean - Louis AYME [1]



Résumé. L'auteur s'intéresse au Problème **2** des 50e *Kürschák Mathematics Competition* et présente une solution synthétique originale, accompagnée d'une note historique, d'une courte biographie ainsi que d'une archive.
Les figures sont toutes en position générale et tous les théorèmes cités peuvent tous être démontrés synthétiquement.

Remerciements. Ils vont tout particulièrement au professeur Ercole Suppa (Italie).
Sa passion pour la Géométrie du Triangle mérite d'être remarquée et soulignée par les Géomètres contemporains.

[1] St-Denis, Île de la Réunion (Océan Indien, France), le 31/03/2018 ; jeanlouisayme@yahoo.fr

Abstract. The author is interested in the Problem **2** of the 50th *Kurschak Mathematics Competition* and presents an original proof with a historical note, a short biography as well as a archive.
The figures are all in general position and all cited theorems can all be proved synthetically.

Aknowledgment. They go particularly to professor Ercole Suppa (Italy).
His passion for the geometry of the Triangle should be noticed and underlined by the contemporary Geometers.

Riassunto. L'autore si interessa al Problema **2** della cinquantesima *Kürschák Mathematics Competition* e presenta una soluzione sintetica originale, accompagnata da una nota storica, da una corta biografia e da un archivio.
Le figure sono tutte in posizione generale e tutti i teoremi citati possono essere dimostrati sinteticamente.

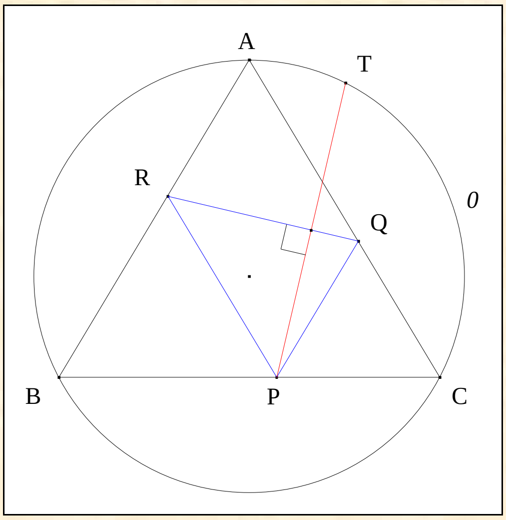
Ringraziamenti. Vanno tutti in particolare al professor Ercole Suppa (Italia)
La sua passione per la Geometria del Triangolo merita di essere notata e sottolineata dai geometri contemporanei.

Sommaire

A. LE PROBLÈME 2

VISION

Figure :



Traits :	ABC	un triangle A-isocèle,
	0	le cercle circonscrit à ABC,
	P	un point de [BC],
	Q	le point d'intersection de la parallèle à (AB) issue de P avec (AC),
	R	le point d'intersection de la parallèle à (AC) issue de P avec (AB)
et	T	le symétrique de P par rapport à (QR).

Donné : T est sur *0*.[2]

Commentaire : une figure élégante qui a attiré le regard de l'auteur.

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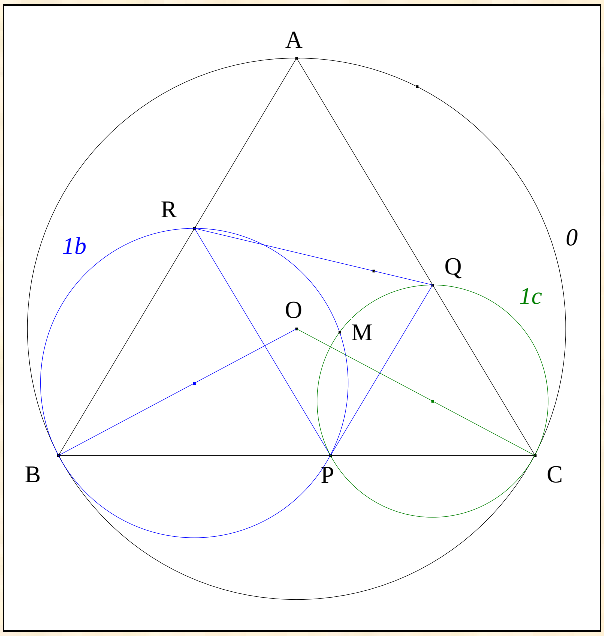
2 50th *Kürschák Mathematics Competition* (1949)
Leigh R. B., Liu A., *Hungarian Problem Book* **IV**, MAA (2011) 45-48
Suppa E., A compilation of the Eötvös- Kürschák Competition (13/11/2007) 60 ;
http://www.batmath.it/matematica/raccolte_es/ek_competitions/ek_competitions.pdf
Tournament of Towns Spring 2015 Senior A-level ; http://www.math.toronto.edu/oz/turgor/archives/TT2015S_SAproblems.pdf
Reflection of a Point lying on Circumcircle, AoPS du 24/02/2017 ;
https://www.artofproblemsolving.com/community/c6t48f6h1389049_reflection_of_a_point_lying_on_circumcircle
Barroso R., Quincena del 16 al 31 de Marzo de 2018, problema **872** ; http://personal.us.es/rbarroso/trianguloscabri/

VISUALISATION

Commentaire : les parallélismes exprimés dans les hypothèses, nous invite au théorème de Reim.



- Le cercle *0*, le point de base B, les moniennes naissantes (CBP) et (ABR), les parallèles (CA) et (PR), conduisent au théorème **7''** de Reim ;
  en conséquence, le cercle circonscrit au triangle BPR est tangent à *0* en B.

- Notons *1b* ce cercle.

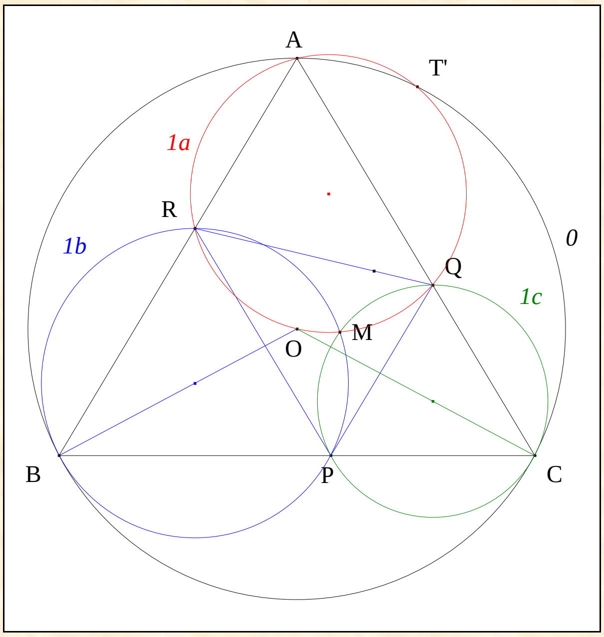
- Mutatis mutandis, nous montrerions que le cercle circonscrit au triangle CPQ est tangent à *0* en C.

- Notons *1c* ce cercle,
  M le second point d'intersection de *1b* et *1c*,
  et O le centre de *0*.

- **Scolies :** **(1)** (OB) est une droite diamétrale de *1b*

  (2) (OC) est une droite diamétrale de *1c*.

Commentaire : un point sur chaque côté d'un triangle, nous invite au théorème du pivot.

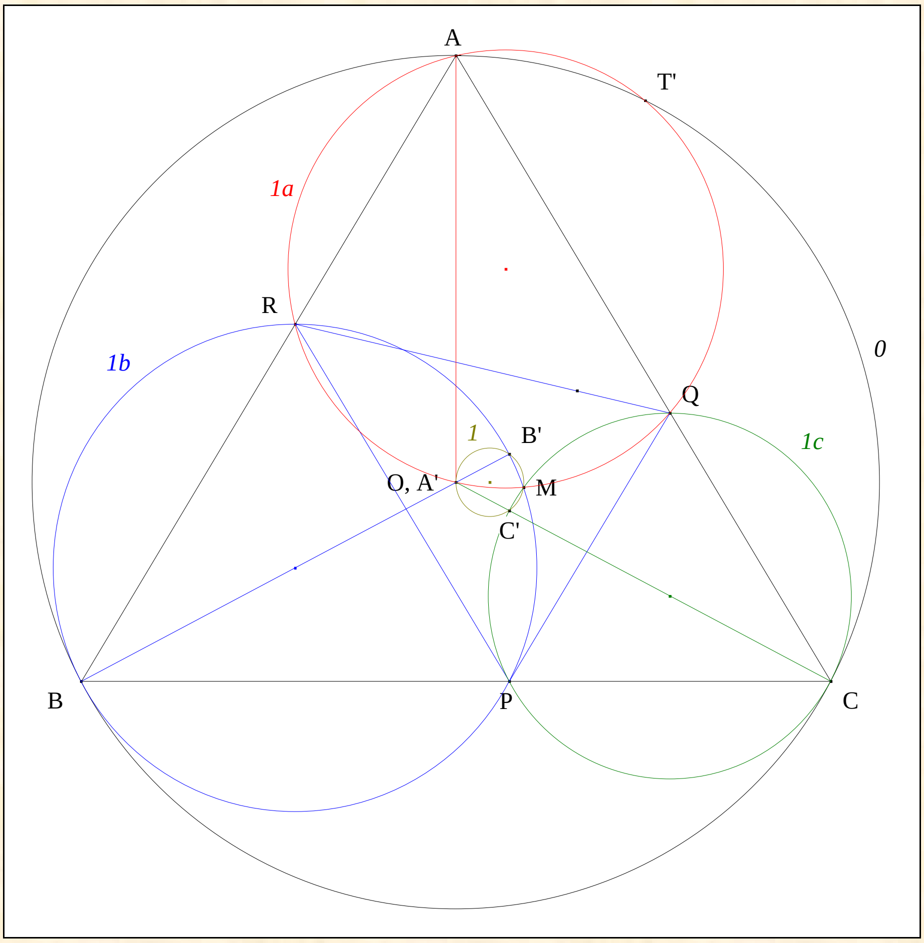


- Notons *1a* le cercle circonscrit au triangle AQR
  et T' le second point d'intersection de *1a* avec *0*.

- D'après Auguste Miquel "Le théorème du pivot" [3], *1a*, *1b* et *1c* concourent en M.

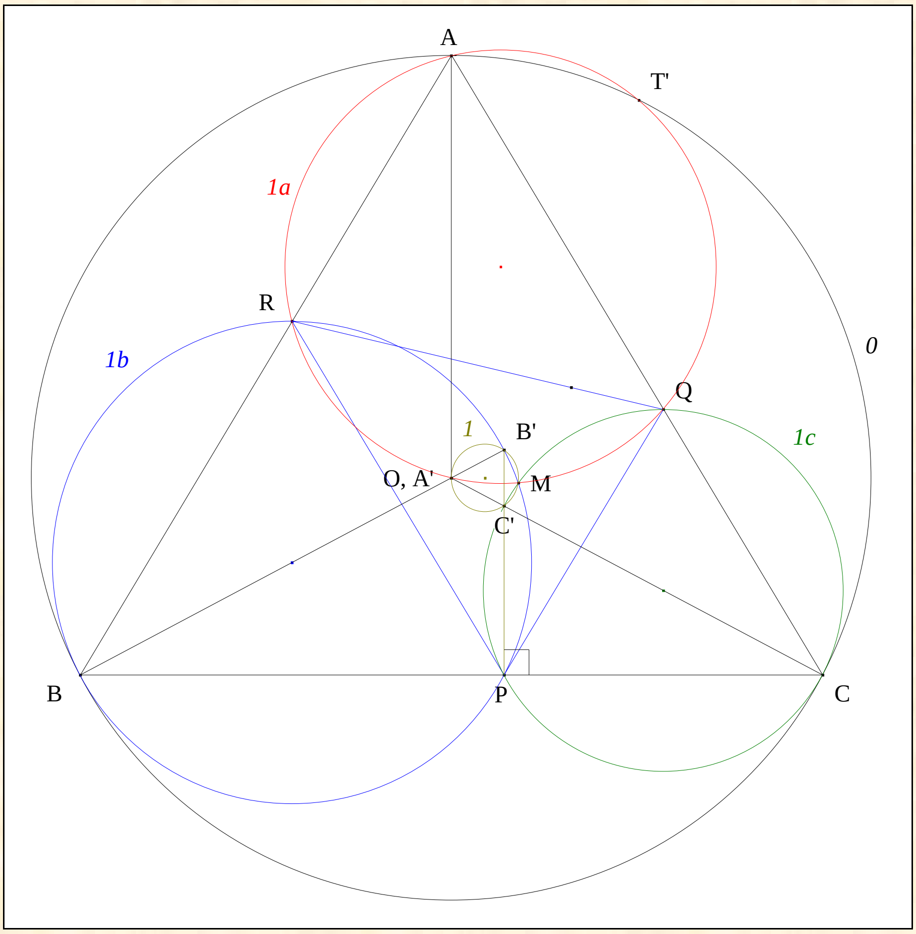
[3] Ayme J.-L., Auguste Miquel, G.G.G. vol. **13**, p. 4-7 ; http://jl.ayme.pagesperso-orange.fr/

Commentaire : nous observons que le cercle *1a* passe par O.



- Notons A', B', C' les seconds points d'intersection de (OA) et *1a*, (OB) et *1b*, (OC) et *1c*.
- D'après "Le cercle de Mannheim" [4], A', B', C', O et M sont cocycliques.
- Notons *1* ce cercle.

[4] Ayme J.-L., Les cercles de Morley, Euler, Mannheim…, G.G.G. vol. **2** ; http://jl.ayme.pagesperso-orange.fr/



- **Scolies :** **(1)** [OB'] est un diamètre de *1b*

  (2) [OC'] est un diamètre de *1c*.

- D'après Thalès "Triangle inscriptible dans un demi-cercle", (PB') ⊥ (BPC) et (BPC) ⊥ (PC') ;
  d'après l'axiome **IVa** des perpendiculaires, (PB') // (PC') ;
  d'après le postulat d'Euclide, (PB') = (PC') ;
  en conséquence, P, B' et C' sont alignés.

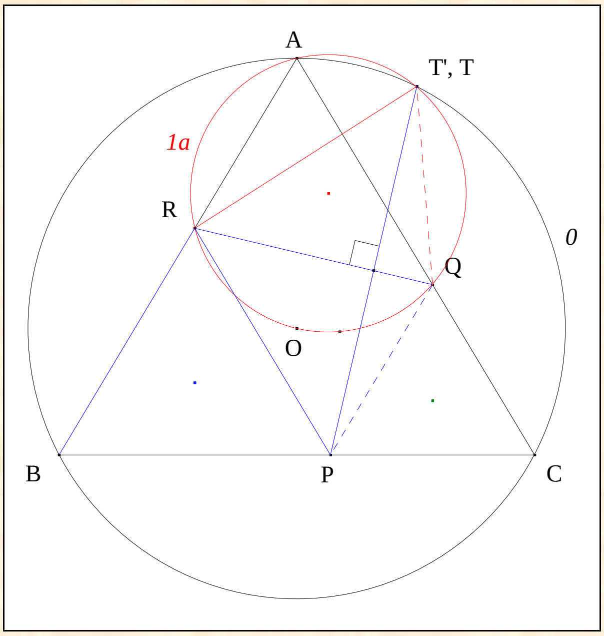
- Le triangle ABC étant A-isocèle, (OA'A) ⊥ (BC) ;
  nous savons que (BC) ⊥ (B'C') ;
  d'après l'axiome **IVa** des perpendiculaires, (OAA') // (B'C').

- Les triangles RBC et QCP étant resp. R, Q-isocèles,
  d'après les théorèmes "Angles inscrits" et "Angles opposés", le triangle OB'C est O-isocèle.

- (OA') étant parallèle à (B'C'), (OA') est la tangente à *1* en O ;
  en conséquence, O et A' sont confondus.

- **Conclusion partielle :** *1a* passe par O.

Commentaire : un résultat peu connu s'impose…



- D'après "Cercle passant par le centre d'un cercle" [5], RT' = RB ;
  le triangle RBP étant R-isocèle, RB = RP ;
  par transitivité de =, RT' = RP.

- Mutatis mutandis, nous montrions que QT' = QP.

- D'après "Le théorème de la médiatrice", (QR) est la médiatrice de [PT'].

- T' étant le symétrique de P par rapport à (QR), T' et T sont confondus.

- **Conclusion :** T est sur *0*.

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[5] Ayme J.-L., Simplicity 1, G.G.G. vol. **36**, p. 9-10 ; http://jl.ayme.pagesperso-orange.fr/

B. LES COMPÉTITIONS KÜRSCHAK

1. Une introduction d'Ercole Suppa

[6]

There are several separate competitions in Hungary. The oldest modern mathematical competition, not only in Hungary but also in the world, is the Kürschák Mathematical Competition, founded in 1894, but known as Eötvös Mathematical Competition until 1938. This competition is for students up to the first year of university and consists of 3 problems. This competition changed its name from Eötvös to Kürschák after the second world war. The Eötvös was not held in the years 1919,1920,1921,1944,1945,1946. The Kürschák was not held in the year 1956. [7]

2. Une courte biographie de József Kürschák [8]

József Kürschák est né le 14 mars 1864 à Buda (Budapest, Hongrie).
Fils de Jozefa Teller et d'András Kürschák, József Kürschák entre en 1881 à l'université de Budapest et en sort en 1886 avec le permis d'enseigner les mathématiques et la physique dans les lycées.
Il enseigne alors à Roznyo (Slovaquie) durant deux années avant de retourner à l'université de Budapest où il obtient son doctorat en 1890 ce qui lui permet d'y rester jusqu'à sa retraite.

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6 Suppa E., Home Page : http://www.esuppa.it
D'Ignazio I, Suppa E., *Il problema geometrico, dal compasso al Cabri*, Interlinea Editrice, Teramos, 2001 ; ISBN 88-85426-16-1

7 Suppa E., A compilation of the Eötvös- Kürschák Competition (13/11/2007) 60 ; http://www.batmath.it/matematica/raccolte_es/ek_competitions/ek_competitions.pdf

8 J J O'Connor and E F Robertson, http://www-history.mcs.st-andrews.ac.uk/Biographies/Kurschak.html

Membre de l'Académie des Sciences en 1897, professeur en titre en 1900, il contribue à l'organisation des compétitions de mathématiques Eötvös Loránd en 1925 qui seront renommées József Kürschák en 1949.
Il décède le 26 mars1933 à Budapest (Hongrie).

Pour l'histoire, rappelons les deux villes de Obuda et Pest sont réunifiées en 1872 sous le nom de Budapest...

3. Archives

Quincena del 16 al 31 de Marzo de 2018

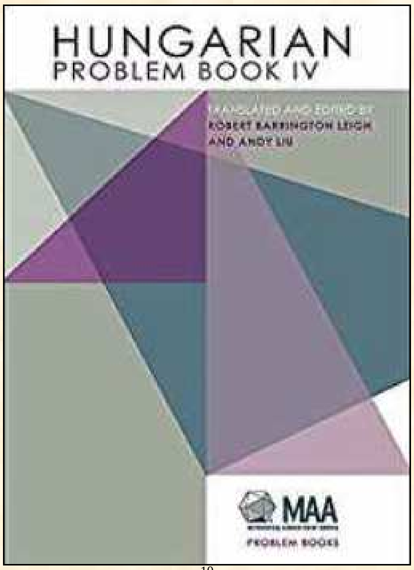
Problema 872

Problema 2. Sea P cualquier punto de la base de un triángulo isósceles. Sean Q y R las intersecciones de los lados iguales con las rectas

que contienen a P paralelas a los mismos. Probar que la reflexión de P según la recta QR pertenece a la circunferencia circunscrita al triángulo dado.

Leigh, R.B., Liu, A. (2011): Hungarian Problem Book IV. MAA.

[9]



[10]

1949

Problem 1. Prove that $\sin A + \frac{1}{2}\sin 2A + \frac{1}{3}\sin 3A > 0$ if $0° < A < 180°$. (Solution is on p. 65.)

Problem 2. Let P be any point on the base of a given isosceles triangle. Let Q and R be the intersections of the equal sides with lines drawn through P parallel to these sides. Prove that the reflection of P about the line QR lies on the circumcircle of the given triangle. (Solution is on p. 45.)

Problem 3. Which positive integers cannot be expressed as sums of two or more consecutive positive integers? (Solution is on p. 34.)

[9] Barroso R., Quincena del 16 al 31 de Marzo de 2018, problema **872** ; http://personal.us.es/rbarroso/trianguloscabri/

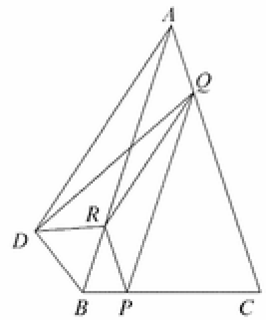
[10] https://books.google.com/books/about/Hungarian_Problem_Book_IV.html?id=_ckF0o53DZUC

First Solution Let ABC be a triangle with $AB = AC$, so that we have $\angle ABC = \angle ACB$. Let P, Q and R be on sides BC, CA and AB respectively. Let D be the point of reflection of P across QR. Then $RP = RD$ and $QP = QD$. Moreover, $AQPR$ is a parallelogram, so that $AR = QP = QD$ and $AQ = PR = DR$. It follows that triangles QAD and RDA are congruent, so that $\angle QAD = \angle RDA$. Since RBP is similar to ABC,

$RB = RP = RD$, so that we have $\angle RBD = \angle RDB$. Hence

$$\begin{aligned}\angle ADB + \angle BCA &= \angle RDA + \angle RDB + \angle ACB \\ &= \angle QAD + \angle RBD + \angle ABC \\ &= \angle CAD + \angle DBC.\end{aligned}$$

Since the sum of all four angles is $360°$, the sum of each pair is $180°$, so that $ADBC$ is a cyclic quadrilateral.



Second Solution Let ABC be a triangle with $AB = AC$. Let P, Q and R be on sides BC, CA and AB respectively. Let D be the point of reflection of P across QR. Then we have

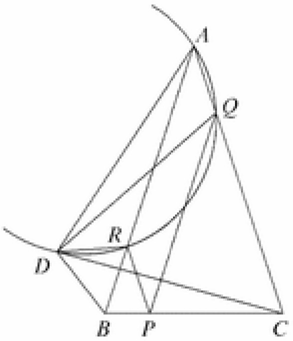
$$\angle CAB = \angle RPQ = \angle RDQ.$$

Hence $AQRD$ is a cyclic quadrilateral. It follows that $\angle AQD = \angle ARD$ and $\angle DQC = \angle DRB$. Note that we have $DQ = PQ = CQ$ and $DR = PR = BR$. Hence the isosceles triangles DQC and DRB are similar, so that $\angle BDR = \angle CDQ$. Now

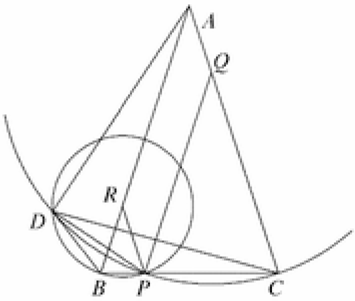
$$\begin{aligned}\angle BDC &= \angle BDQ - \angle CDQ \\ &= \angle BDQ - \angle BDR \\ &= \angle RDQ \\ &= \angle BAC.\end{aligned}$$

Since A and D are on the same side of BC, $ADBC$ is a cyclic quadrilateral.

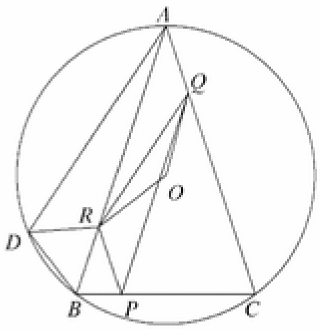
Third Solution Let ABC be a triangle with $AB = AC$. Let P, Q and R be on sides BC, CA and AB respectively. Let D be the point of reflection of P across QR. Then $RP = RD$ and $QP = QD$. Moreover, both RBP



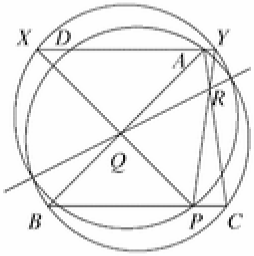
and QPC are similar to ABC. Hence $RB = RP = RD$ and $QC = QP = QD$. It follows that R is the circumcenter of DBP and Q is the circumcenter of DPC. We now have $2\angle CDP = \angle CQP$ and $2\angle PDB = \angle BRP$. Since $\angle CQP = \angle CAB = \angle BRP$, $\angle CDB = \angle CDP + \angle PDB = \angle CAB$. Since A and D are on the same side of BC, $ADBC$ is a cyclic quadrilateral.



Fourth Solution Let ABC be a triangle with $AB = AC$. Let P, Q and R be on sides BC, CA and AB respectively. Let D be the point of reflection of P across QR. Then $RP = RD$ and $QP = QD$. Moreover, $AQPR$ is a parallelogram, so that $AR = QP = QD$ and $AQ = PR = DR$. It follows that triangles QAD and RDA are congruent, so that $ADRQ$ is an isosceles trapezoid. Let O be the circumcenter of ABC. Rotate the segment AB about O until the image of A coincides with C and the image of B coincides with A. Then the image of R coincides with Q. It follows that $OQ = OR$, so that O lies on the perpendicular bisector of QR. Hence O also lies on the perpendicular bisector of AD, so that D indeed lies on the circumcircle of ABC.



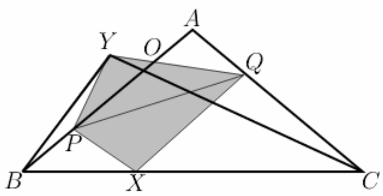
Fifth Solution We prove a more general result, where ABC is an arbitrary triangle. P is any point on BC. Q and R are points on CA and AB such that $QP = QC$ and $RP = RB$, respectively. Let the line through A parallel to BC cut the extensions of PQ and PR at Y and X, respectively. Then we have $\angle CAY = \angle ACP = \angle YPC$ and $\angle BAX = \angle ABP = \angle XPB$. It follows that $\angle CAB = \angle YPX$. Now $AC = PY$ and $AB = PX$. Hence triangles ABC and PXY are congruent, so that they have equal circumradii. The power of the point Q with respect to the two circles are $QA \cdot QC = QY \cdot QP$. Hence it lies on the radical axis of the two circles. Similarly, so does R. Since the two circles are congruent, they are symmetric about QR, and the point D which is the image of P across QR lies on the circumcircle of ABC.



Tournament of Towns in Toronto (2015) [11]

Problem 2. A point X is marked on the base BC of an isosceles triangle ABC, and points P and Q are marked on the sides AB and AC so that $APXQ$ is a parallelogram. Prove that the point Y symmetrical to X with respect to line PQ lies on the circumcircle of the triangle ABC.

Solution (Richard Chow). Consider triangles PYO and OAQ. Note that $\angle PYQ = \angle PXQ = \angle PAQ$ and $\angle YOP = \angle AOQ$. Then $\angle YPO = \angle AQO$ which implies that $\angle BPY = \angle YQC$. Since triangle ABC is isosceles and $PX \parallel AC$, triangle BPX is also isosceles and since $PX = PY$, triangle BPY is isosceles as well. In similar way we can rove that triangle YQC is also isosceles. Then triangles BPY and YQC are similar. If follows that $\angle BYC = \angle BAC$ and therefore quadrilateral $BYAC$ is cyclic.



[12]

[11] http://www.math.toronto.edu/oz/turgor/archives.php

[12] http://www.math.toronto.edu/oz/turgor/archives/TT2015S_SAsolutions.pdf