Problema 784 de trianguloscabri

3609. *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*

Let r be a real number. and let D, E, and F be points on the sides BC, CA, and AB of a triangle ABC with

$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = r.$$

The cevians AD, BE, and CF bound a triangle PQR whose area we denote by $[PQR]$. Find the value of r for which the ratio of the areas, $\frac{[DEF]}{[PQR]}$ equals 4.

Solution of Ricardo Barroso Campos.

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Let a=BC, b=AC, c=AB.

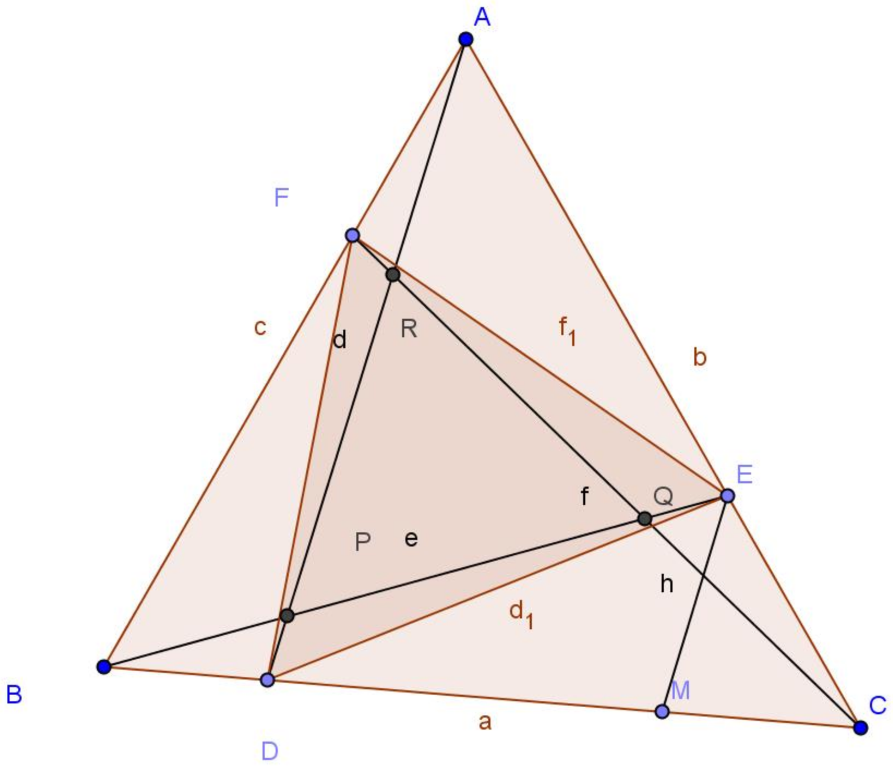
Is: $BD = \frac{ar}{1+r}$, $= \frac{a}{1+r}$, $CE = \frac{br}{1+r}$, $EA = \frac{b}{1+r}$, $AF = \frac{cr}{1+r}$, $FB = \frac{c}{1+r}$

Is: $[CED] = \frac{CE}{CA}\frac{CD}{CB}[ABC] = \frac{{br}/{(1+r)}}{b}\frac{{a}/{(1+r)}}{a}[ABC] = \frac{r}{(1+r)^2}[ABC]$

And [CED]=[BDF]=[AFE]

Is $[DEF] = [ABC] - 3[CED] = \frac{1-r-r^2}{(1+r)^2}[ABC]$

$$[BEC] = \frac{r}{1+r}[ABC], [CAD] = \frac{1}{1+r}[ABC]$$



Let M on BC such that EM is parallel to AD

$\frac{CM}{CE} = \frac{CD}{CA}, CM = \frac{CD\,CE}{CA} = \frac{\left(a/_{(1+r)}\right)\left(br/_{(1+r)}\right)}{b} = \frac{ar}{(1+r)^2}$

$$BM = a - CM = \frac{a(1+r+r^2)}{(1+r)^2}$$

$$[ECM] = \frac{CM}{CD}\frac{CE}{CA}[CAD] = \frac{ar/_{(1+r)^2}}{a/_{(1+r)}}\frac{br/_{(1+r)}}{b}[CAD] = \frac{r^2}{(1+r)^2}[CAD]$$

Then is $[ECM] = \frac{r^2}{(1+r)^3}[ABC]$

$$[BEM] = [BEC] - [ECM] = \frac{r}{1+r}[ABC] - \frac{r^2}{(1+r)^3}[ABC] = \frac{r+r^2+r^3}{(1+r)^3}[ABC]$$

Let P intersection of AD with BE.

The triangle BDP is homotetic of BME, with ratio AD/AM.

Is: $\frac{[BDP]}{[BME]} = \frac{BD^2}{BM^2} \gg [BDP] = \frac{\left(ar/_{1+r}\right)^2}{\left(a(1+r+r^2)/_{(1+r)^2}\right)^2}\frac{(r+r^2+r^3)}{(1+r)^3}[ABC]$

Or $[BDP] = \frac{r^3}{(1+r)(1+r+r^2)}\ [ABC]$

Is [BDP]=[ARF]=[CQE].

Is $[DPQC] = [BEC] - [BDP] - [ECQ] = \frac{r}{(1+r)}[ABC] - \frac{2r^3}{(1+r)(1+r+r^2)}\ [ABC]$

Or $[DPQC] = \frac{r+r^2-r^3}{(1+r)(1+r+r^2)}\ [ABC]$

Then, [DPQC]=[ARQE]=[BPRF].

Is: [PQR]=[ABC]- 3[BPD]-3[DPQC]=

$$[\mathrm{PQR}] = [\mathrm{ABC}] - 3[\mathrm{BPD}] - 3[\mathrm{DPQC}] = \left(1 - 3\frac{r^3}{(1+r)(1+r+r^2)} - 3\frac{r+r^2-r^3}{(1+r)(1+r+r^2)}\right)[ABC]$$

And

$$[\mathrm{PQR}] = \frac{1-2\mathrm{r}+\mathrm{r}^2}{1+\mathrm{r}+\mathrm{r}^2}[ABC]$$

Then, [EDF]=4 [PQR], is:

$$\frac{1-r+r^2}{(1+r)^2}[ABC] = 4\frac{1-2r+r^2}{1+r+r^2}[ABC]$$

Simplifying, $r^4 - 3r^2 + 1 = 0,, r = \sqrt{\frac{3\pm\sqrt{5}}{2}}$

r =ϕ, r=1/ϕ.

Esta solución se reseña en Crux Mathematicorum (pag. 39-40) en el número de Enero de 2012