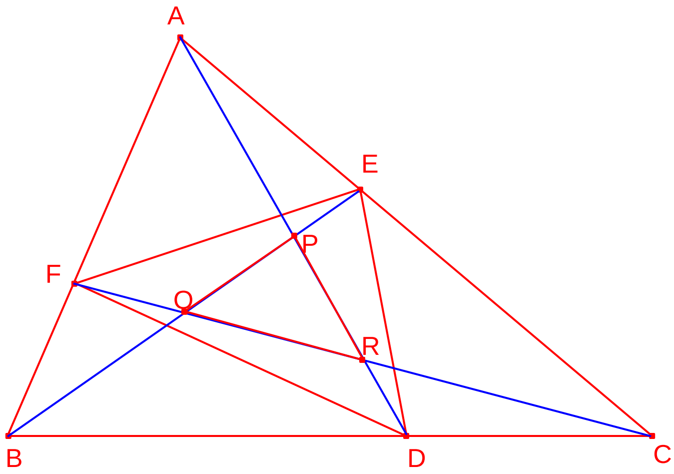
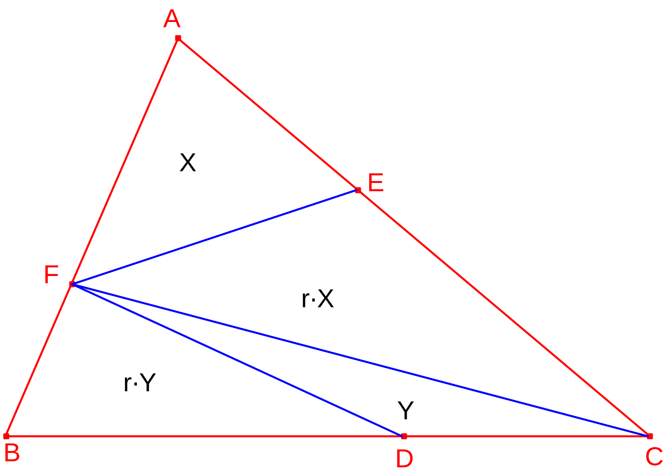
Problema 874

3609. Sea r un número real y D, E y F puntos sobre los lados BC, CA y AB de un triángulo ABC tal que BD/DC=CE/EA=AF/FB=r. Las cevianas AD, BE, y CF limitan un triángulo PQR cuya área es [PQR] . Hallar el valor de r para el cual la relación de áreas [DEF]/[PQR] es 4.

Ligouras, P. (2011): Crux Mathematicorum (37-1). Pg. 50

Solución de Ricard Peiró:





Sea $S = S_{ABC}$

Sea $X = S_{AEF}$, $Y = S_{DCF}$.

$S_{CEF} = rX$, $S_{BDF} = rY$.

$(1+r)X = r(1+r)Y$.

Entonces, $Y = \frac{1}{r}X$.

$(1+r)(X+Y) = S$.

$X = \frac{r}{(r+1)^2}S$.

Análogamente, $S_{BDF} = S_{CDE} = X = \frac{r}{(r+1)^2}S$.

$S_{DEF} = S - 3X = \frac{r^2 - r + 1}{(r+1)^2}S$.

Sea $U = S_{BFQ}$, $V = S_{AEQ}$.

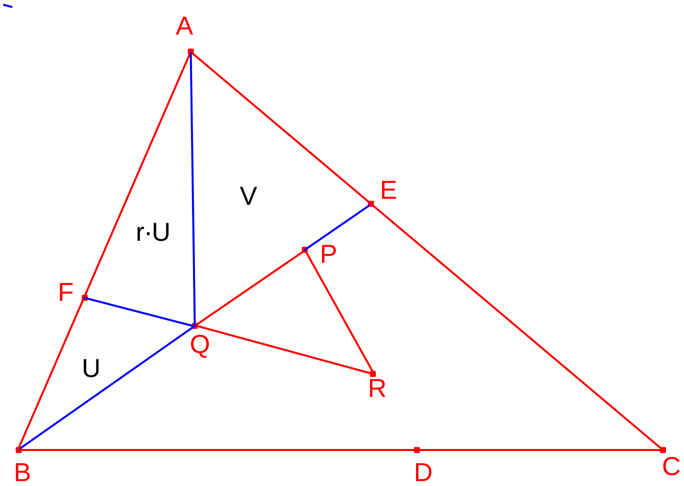
$S_{AFQ} = rU$, $S_{ECQ} = rV$.

$S_{AFC} = \dfrac{r}{r+1} S$.

$rU + (r+1)V = \dfrac{r}{r+1} S$ (1)

$S_{ABE} = \dfrac{1}{r+1} S$

$(r+1)U + V = \dfrac{1}{r+1} S$ (2)



Resolviendo el sistema formado por las expresiones (1) (2):

$$\begin{cases} U = \dfrac{1}{(r+1)(r^2+r+1)} S \\ V = \dfrac{r^2}{(r+1)(r^2+r+1)} S \end{cases}$$

Análogamente, $S_{APE} = S_{DCR} = U = \dfrac{1}{(r+1)(r^2+r+1)} S$.

$S_{AFQP} = S_{AFQ} + S_{AQE} - S_{APE} = (r-1)U + V = \dfrac{r^2+r-1}{(r+1)(r^2+r+1)} S$.

Análogamente, $S_{BDRQ} = S_{CEPR} = S_{AFQP} = \dfrac{r^2+r-1}{(r+1)(r^2+r+1)} S$.

$S_{PQR} = S - 3U - 3S_{AFQP} = \left(1 - \dfrac{3}{(r+1)(r^2+r+1)} - 3\dfrac{r^2+r-1}{(r+1)(r^2+r+1)}\right) S$.

$S_{PQR} = \dfrac{r^3 - r^2 - r + 1}{(r+1)(r^2+r+1)} S$.

$4 \cdot S_{PQR} = S_{DEF}$.

Entonces,

$4\dfrac{r^3 - r^2 - r + 1}{(r+1)(r^2+r+1)} = \dfrac{r^2 - r + 1}{(r+1)^2}$.

Simplificadot:

$r^4 - 3r^2 + 1 = 0$.

Resolviendo la ecuación:

$r = \sqrt{\dfrac{3+\sqrt{5}}{2}}$, $r = \sqrt{\dfrac{3-\sqrt{5}}{2}}$.