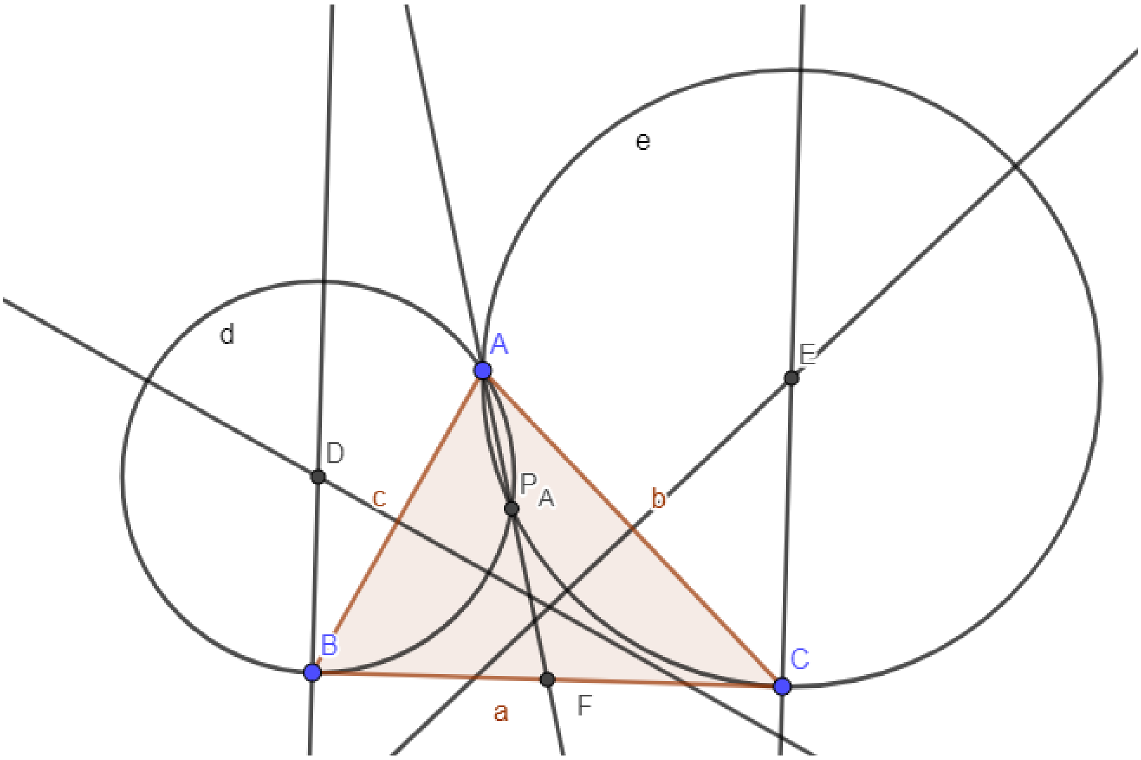
12073. *Proposed by Hakan Karakus, Antalya, Turkey.* Given a scalene triangle ABC, let G denote its centroid and H denote its orthocenter. Let P_A be the second point of intersection of the two circles through A that are tangent to BC at B and at C. Similarly define P_B and P_C. Prove that G, H, P_A, P_B, and P_C are concyclic.

Solution of Ricardo Barroso Campos. Sevilla. Spain



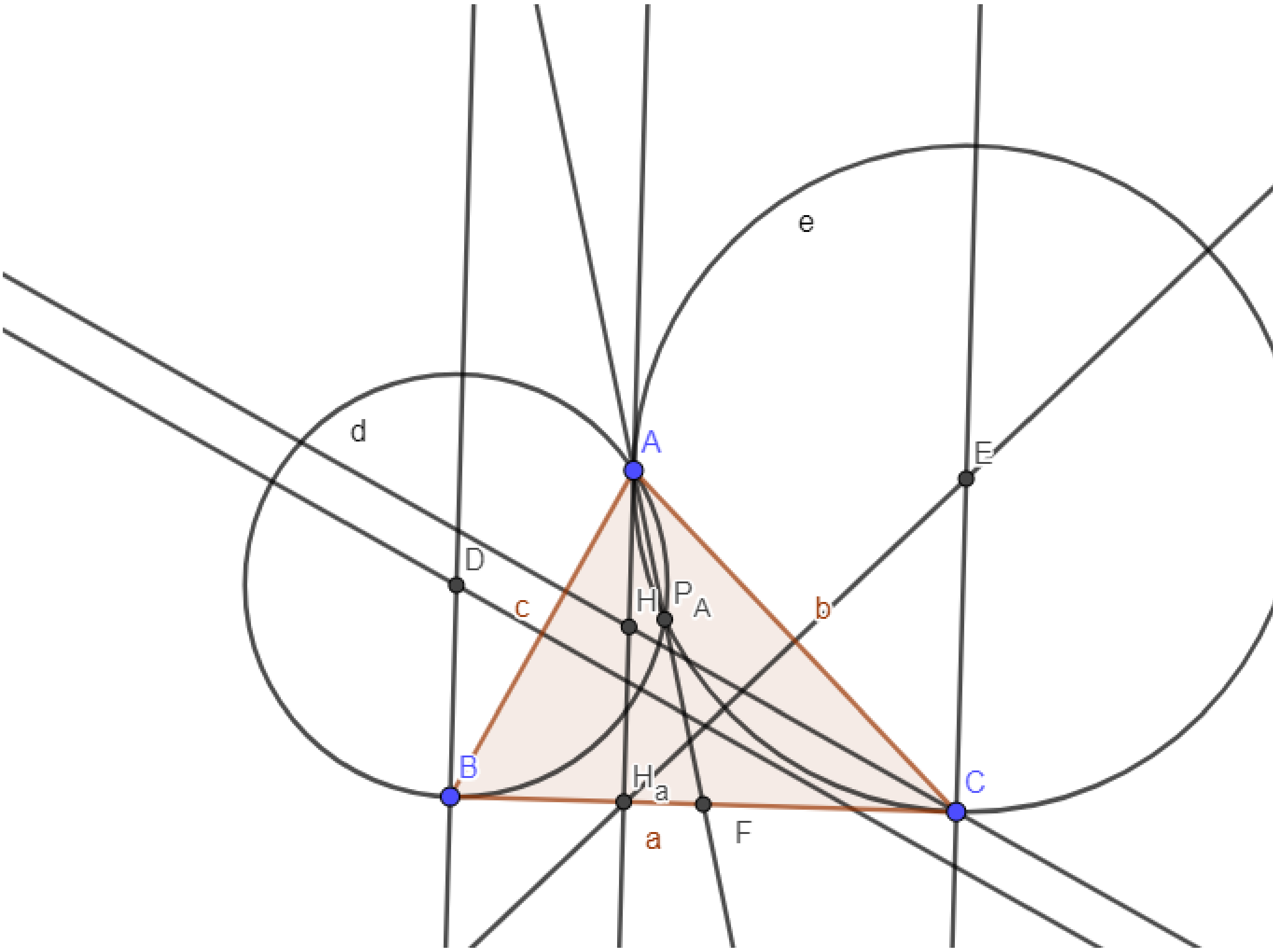
Let F the point intersection of lines AP_A and BC.

Is $(FP_A)(FA)=FB^2=(BC-FB)^2$, then FB=FC, and AF is median of ABC. Thus G, the centroide, is on AF.

$$m^2 = AF^2 = \frac{2\,c^2 + 2\,b^2 - a^2}{4}$$

$(FP_A)(FA)= (x\,FA)(FA)= \frac{a^2}{4}$; $x=\frac{a^2}{4\,m^2}$

Thus $AP_A=AF-P_AF= m-xm= \frac{4m^2-a^2}{4m}$



Let H orthocencer. The four points H H_a F and P_A are concyclic.

$(AP_A)(AF) = \frac{4m^2-a^2}{4m}\, m = \frac{4m^2-a^2}{4} = \frac{4\frac{2c^2+2b^2-a^2}{4}-a^2}{4} = \frac{c^2+b^2-a^2}{2}$

$AH_a = c \sin \beta$

$AH = \frac{b\cos\alpha}{\sin\beta}$

$(AH)\,(AH_a) = bc\cos\alpha = \frac{c^2+b^2-a^2}{2}$

Thus, $\angle HP_A G = \angle HP_A F = \angle HH_a F = 90º$

Similarly, $\angle HP_B G = \angle HP_C G = 90º$

Thus G, H, P_A P_B P_C are concyclic.