Proposal for publication

Mihaela Berindeanu, teacher, Bucharest, România

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Let ABC be an acute triangle, with circumcenter O and orthocenter H. Let A', B', C' be the intersection of AH, BH, CH with BC, AC, AB and A_1, B_1, C_1 be the intersection of AO, BO, CO with BC, AC, AB. If A'', B'', C'' are AA_1, BB_1, CC_1 midpoints, show that $A'A''$, $B'B''$, $C'C''$ have a common intersection point.

Solution:
Notations: $\triangle ABC$ circumcircle $= \Gamma$, $AH \cap \Gamma = \{X\}$, $AO \cap \Gamma = \{Y\}$, $E \in A'A''$, $A'E = EA''$
Remarks:
• $X \in \Gamma$ is the orthocenter's symmetrical point towards $BC \Rightarrow HA' = A'X$;
• Y is the symmetrical point towards the circumcenter $\Rightarrow OA = OY$.

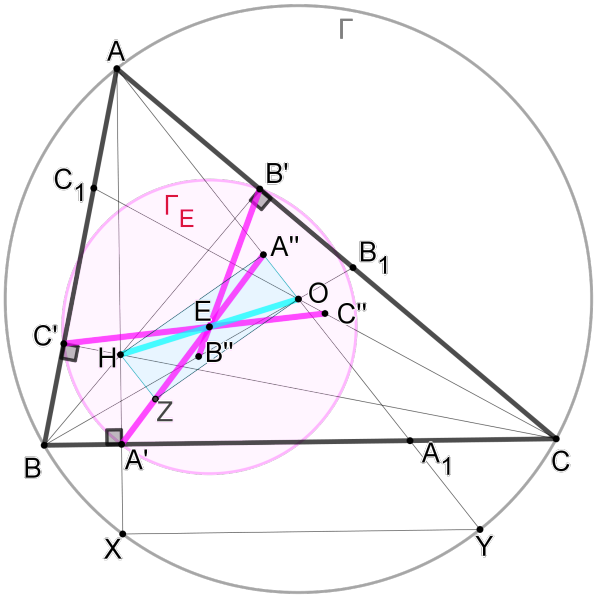

Figura 1:

- Prove that $A'A''$ cut OH in E and E is the center of $\triangle ABC$ nine-point circle.
  $HA' = A'X$ (because $X \in \Gamma$ is the orthocenter's symmetrical point towards BC)
  AY is the circumcircle's diameter of $\triangle ABC \Rightarrow \measuredangle AA'A_1 = AXY = 90° \Rightarrow XY \parallel A'A_1$, so, in $\triangle AXY$:

$$\frac{XA'}{AA'} = \frac{YA_1}{AA_1} \Rightarrow \frac{HA'}{AA'} = \frac{AY - AA_1}{AA_1} = \frac{2AO - 2AA''}{2AA''} = \frac{AO - AA''}{AA''} = \frac{OA''}{AA''}$$

  Let $Z \in A'A''$, $HZ \parallel AY \Rightarrow$ in $\triangle AA'A''$, $HZ \parallel A'A''$, so:

$$\frac{HA'}{AA'} = \frac{HZ}{AA''}$$

$$\frac{HA'}{AA'} = \frac{OA''}{AA''} = \frac{HZ}{AA''} \Rightarrow OA'' = HZ$$

  From $HZ \parallel A''O$ and $HZ = A''O \Rightarrow HZOA''$ is a parallelogram $\Rightarrow HO \cap A'A'' = \{E\}$ and $EH = EO$, so E is the center of $\triangle ABC$ nine-point circle.

- Analogously is proved that

$$B'B'' \cap HO = E$$

$$C'C'' \cap HO = E$$

Conclusion:

$$A'A'' \cap B'B'' \cap C'C'' = \{E\}$$