Proposal for publication

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Let ABC be a triangle with orthocenter H, circumcircle Γ and circumcenter O. Denote the midpoints of AB, respectively AC with C' respectively B', the circumcenters of ABB' respectively ACC' with Γ_1, respectively Γ_2. The diametrically opposite points of A in circles Γ_1 respectively Γ_2 are B_1, respectively C_1. Prove that $\overrightarrow{BB_1} + \overrightarrow{CC_1} + \overrightarrow{OH} = \overrightarrow{0}$

<u>Solution:</u>
Denote the midpoints of AB_1, respectively AC_1 with B'', respectively C''

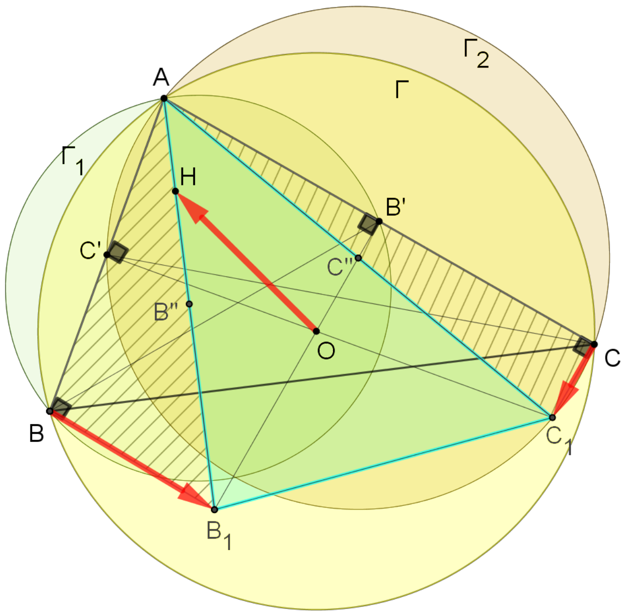

Figura 1:

- Show that $B',\ C'',\ B_1$, respectively $C',\ B'',\ C_1$ are collinear points

$$\left.\begin{array}{l} AB_1 = \text{ diameter of } \Gamma_1 \\ AB'' = B''B_1 \end{array}\right\} \Rightarrow B'' = \text{ circumcenter of } ABB_1 \text{ circumcircle} \Rightarrow$$

Midpoint of AB is $C' \Rightarrow B''C'$ is the perpendicular bisector for AB so,

$$\measuredangle\left(AC'B''\right) = 90^\circ \quad (^*1)$$

AC_1 is diameter for Γ_2 so

$$\measuredangle\left(AC'C_1\right) = 90^\circ \quad (^*2)$$

$$(^*1), (^*2) \Rightarrow C',\ B'',\ C_1 \text{ are collinear points}$$

Analogously

$$B',\ C'',\ B_1 \text{ are collinear points}$$

- Show that O is the centroid of $\triangle\ AB_1C_1$
$B''C'$ respectively $C''B'$ are perpendicular bisectors for AB, respectively AC, so

$$C_1C' \cap B_1B' = \{O\}$$

In $\triangle\ AB_1C_1,\ B_1C''$ and C_1B'' are medians so O is the centroid of $\triangle\ AB_1C_1$

- Show that $\overrightarrow{BB_1}+\overrightarrow{CC_1}+\overrightarrow{OH}=\overrightarrow{0}$
  $O=$ the centroid of $\triangle\ AB_1C_1$, so:

$$\overrightarrow{OA}+\overrightarrow{OB_1}+\overrightarrow{OC_1}=\overrightarrow{0} \qquad (^*3)$$

  Applying Sylvester's relationship in the $\triangle\ ABC$

$$\overrightarrow{OH}=\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}\Rightarrow\overrightarrow{OA}=\overrightarrow{OH}-\overrightarrow{OB}-\overrightarrow{OC}$$

$$\overrightarrow{OA}=\overrightarrow{OH}+\overrightarrow{BO}+\overrightarrow{CO} \qquad (^*4)$$

  From $(^*3)\,,(^*4)$ results

$$\overrightarrow{OH}+\overrightarrow{BO}+\overrightarrow{CO}+\overrightarrow{OB_1}+\overrightarrow{OC_1}=\overrightarrow{0}$$

$$\overrightarrow{OH}+\underbrace{\left(\overrightarrow{BO}+\overrightarrow{OB_1}\right)}_{\overrightarrow{BB_1}}+\underbrace{\left(\overrightarrow{CO}+\overrightarrow{OC_1}\right)}_{\overrightarrow{CC_1}}=\overline{0}$$

  Finally

$$\overrightarrow{OH}+\overrightarrow{BB_1}+\overrightarrow{CC_1}=\overrightarrow{0}$$