Proposal for publication

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Let ABC be a triangle with circumcircle Γ, circumcenter O and orthocenter H. Denote M the small arc $\widehat{BC}$ of the circle Γ. The parallel through O to MB, respectively MC cuts AB, respectively AC in E, respectively F and the perpendicular bisector of EF cuts the small arc $\widehat{BC}$ of Γ in P. Show that $CP = CH$.

<u>Solution:</u>
Denote $\measuredangle ABC = \beta$, $\Gamma_1 = \triangle\, AEF$ circumcircle, $\Gamma \cap AH = \{S\}$ and $AH \cap \Gamma_1 = \{T\}$.

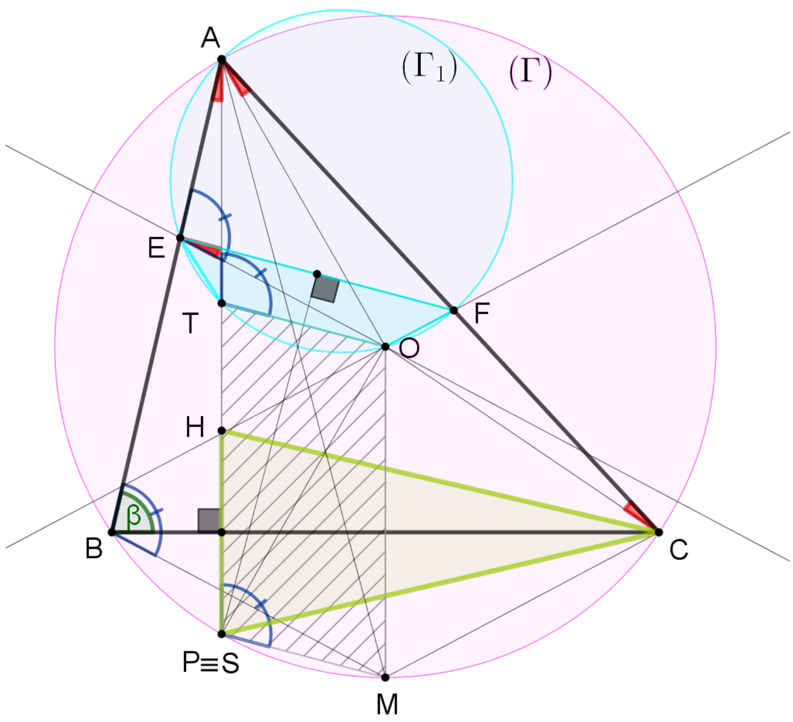

Figura 1:

- Show that $ETOF$ is an isosceles trapezoid
  According to the inscribed angle theorem, in the circle Γ:

$$\left.\begin{array}{l}\angle AOC \text{ central angle} \\ \beta \text{ inscribed angle}\end{array}\right\} \Rightarrow \measuredangle AOC = 2\beta$$

$\triangle\, AOC$ is an isosceles triangle, with

$$\measuredangle OAC = \measuredangle OAF = \frac{180^\circ - 2\beta}{2} = 90^\circ - \beta \qquad (^*1)$$

$$AH \perp BC \Rightarrow \measuredangle BAH = \measuredangle EAH = 90^\circ - \beta \qquad (^*2)$$

From $(^*1), (^*2)$ results

$$\measuredangle OAF = \measuredangle EAH$$

In the circle Γ_1

$$\measuredangle OAF = \measuredangle OEF = \frac{\widehat{OF}}{2}$$

At congruent angles correspond congruent chords, so

$$OF = ET \Rightarrow EF \parallel TO \Rightarrow ETOF \text{ is an isosceles trapezoid}$$

- Show that $TSMO$ is a parallelogram, so $OM = TS$

$$\left.\begin{array}{l}\angle AEO,\ \angle ATO \text{ are inscribed angles in } \mathit{\Gamma}_1\\ AO \text{ common chord for } \angle AEO,\ \angle ATO\end{array}\right\} \Rightarrow \measuredangle AEO = \measuredangle ATO \qquad (^*3)$$

$$EO \parallel BM \Rightarrow \measuredangle AEO = \measuredangle MBA \qquad (^*4)$$

$$\left.\begin{array}{l}\angle MBA,\ \angle MSA \text{ are inscribed angles in } \mathit{\Gamma}\\ AM \text{ common chord for } \angle MBA,\ \angle MSA\end{array}\right\} \Rightarrow \measuredangle MBA = \measuredangle MSA \qquad (^*5)$$

From $(^*3)\,,(^*4)\,,(^5)$ results

$$\measuredangle ATO = \measuredangle MSA \Rightarrow OT \parallel MP$$

$$\left.\begin{array}{l}AH \perp BC\\ OM \perp BC\end{array}\right\} \Rightarrow OM \parallel AH \Rightarrow OTSM = \text{ parallelogram } \Rightarrow OM = TS$$

- Show that $CP = CH$

$$\left.\begin{array}{l}OM = TS\\ OM = OS \text{ radii of } \mathit{\Gamma}\end{array}\right\} \Rightarrow OS = TS$$

S is in the perpendicular bisector of OT and also in the perpendicular bisector of EF (because $ETOF$ is an isosceles trapezoid), so

$$P \equiv S$$

P is the symmetrical reflection of point H against BC, so BC is the perpendicular bisector of HP and

$$\triangle\ HPC \text{ is isosceles triangle } \Rightarrow CP = CH$$