

Triángulos Cabri 1054

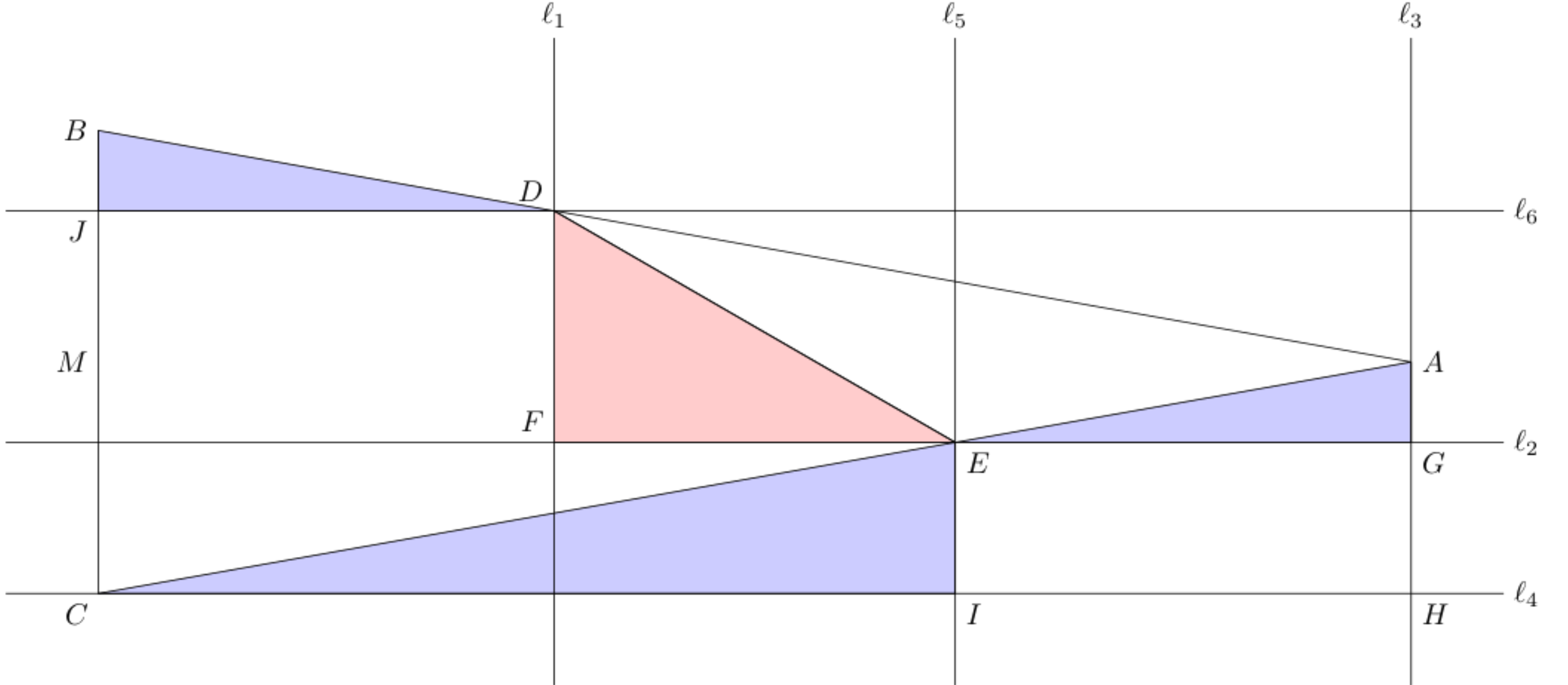
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1 Finding an equation

1.1 Definition

Define the line passing point α parallel to line β as $\{\alpha, \beta\}$. Define the intersection of lines α and β as $[\alpha, \beta]$. Let $\ell_1 = \{D, BC\}$. Let M the midpoint of BC . Let $\ell_2 = \{E, MA\}$. Let $F = [\ell_1, \ell_2]$. Since $BC \perp MA$, $DF \perp FE$ so DFE is a right triangle. Let $\ell_3 = \{A, BC\}$. Let $G = [\ell_2, \ell_3]$. Let $\ell_4 = \{C, MA\}$. Let $H = [\ell_3, \ell_4]$. Then CGA is a right triangle. Let $\ell_5 = \{E, BC\}$. Let $I = [\ell_4, \ell_5]$. Let $\ell_6 = \{D, MA\}$. Let $J = [\ell_5, \ell_6]$. When we are done, we get the picture below.



1.2 Finding lengths

We use the fact that all the blue right triangles are AA similar to find the length of DF and FE . As seen on AGC the

ratio of the length of the sides is $b : \frac{a}{2} : \sqrt{b^2 - \frac{a^2}{4}}$, hypotenuse, height and base respectively.

$$DF = BC - BJ - EI = a - \frac{a}{2} \cdot \frac{a}{b} - \frac{a}{2} \cdot \frac{b-a}{b} = \frac{a}{2}.$$

Meanwhile $FE = CG - JD - EG = \sqrt{b^2 - \frac{a^2}{4}} \cdot \left(1 - \frac{a}{b} - \frac{a}{b}\right) = \sqrt{b^2 - \frac{a^2}{4}} \cdot \frac{b - 2a}{b}$. Now use the Pythagorean theorem on DFE .

$$DF^2 + FE^2 = DE^2 \text{ so } \frac{a^2}{4} + \left(b^2 - \frac{a^2}{4} \cdot \frac{b^2 - 4ab + 4a^2}{b^2}\right) = a^2. \text{ After organizing we get } b^4 - 4b^3a + 3b^2a^2 + ba^3 - a^4 = 0.$$

Dividing by b^4 and substituting $r = \frac{a}{b}$, the equation becomes $r^4 - r^3 - 3r^2 + 4r - 1 = 0$. Since $0 < a < b$, $0 < \frac{a}{b} = r < 1$. Since the LHS equals zero when plugging in $r = 1$, according to the remainder theorem LHS has $(r - 1)$ as a factor. Using that we factor the equation and get $(r - 1)(r^3 - 3r + 1) = 0$. $r = 1$ is not a valid answer given the range of r so we can safely disregard it. Let $f(r) = r^3 - 3r + 1$.

2 Solving the equation

Let's solve $f(r) = 0$. Since $f(-2)f(-1)$, $f(0)f(1)$ and $f(1)f(2)$ are all negative, according to the intermediate value theorem $f(r)$ has one real solution in each of these ranges: $(-2, -1)$, $(0, 1)$, $(1, 2)$. We are looking for the solution in $(0, 1)$.

2.1 Trigonometry

Use the identity $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$. Then $\cos 2\theta = \cos(\theta + \theta) = \cos^2 \theta - \sin^2 \theta = 2\cos^2 \theta - 1$.
 $\cos 3\theta = \cos(\theta + 2\theta) = \cos \theta \cos 2\theta - \sin \theta \sin 2\theta = \cos \theta (2\cos^2 \theta - 1) - 2\sin^2 \theta \cos \theta = \cos \theta (2\cos^2 \theta - 1) - 2(1 - \cos^2 \theta) \cos \theta = 4\cos^3 \theta - 3\cos \theta$.

2.2 Substitution

Let $r = 2\cos \theta$. Then $f(r) = r^3 - 3r + 1 = 2(4\cos^3 \theta - 3\cos \theta) + 1 = 0$. According to the identity we found above
 $2(4\cos^3 \theta - 3\cos \theta) + 1 = 2\cos 3\theta + 1 = 0$. So $\cos 3\theta = -\frac{1}{2}$.

2.3 Solution

Solving $\cos 3\theta = -\frac{1}{2}$ yields $3\theta = 2n\pi \pm \frac{2}{3}\pi$, where n is an integer. Then $\theta = \frac{6n\pi \pm 2\pi}{9}$. Substitute back into $r = 2\cos \theta$ and get $r = 2\cos \frac{6n\pi \pm 2\pi}{9}$. $\frac{6n\pi}{9} = \frac{2n\pi}{3}$ has a cycle of length 3, so we only need to try three values, $n = 0, n = 1, n = 2$.

Comparing all six cases gives three unique solutions, $r = 2\cos \frac{2\pi}{9} = 2\cos \frac{-2\pi}{9}$ or $r = 2\cos \frac{8\pi}{9} = 2\cos \frac{10\pi}{9}$ or $r = 2\cos \frac{14\pi}{9} = 2\cos \frac{4\pi}{9}$. Since $0 < \cos \frac{4\pi}{9} < \cos \frac{3\pi}{9} = \frac{1}{2}$, it is the solution we are looking for. In conclusion, $\frac{b}{a} = r = 2\cos \frac{4\pi}{9} = 2\sin \left(\frac{\pi}{2} - \frac{4\pi}{9}\right) = 2\sin \frac{\pi}{18}$.